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EINSTEIN METRICS ON 5-DIMENSIONAL SEIFERT BUNDLES

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The aim of this paper is to study the existence of positive Ricci curvature metrics, and especially Einstein metrics, on 5-dimensional Seifert bundles.

Seifert fibered 3-manifolds were introduced and studied in [Sei32]. These are 3-manifolds M which admit a differentiable map $f : M \rightarrow F$ to a surface F such that every fiber is a circle. There are finitely many points $p_1, \dots, p_n \in F$ such that M is a circle bundle over $F \setminus \{p_1, \dots, p_n\}$, and the fibers over p_i naturally appear with some multiplicity m_i . It is the *orbifold Euler characteristic*

$$e^{orb}(F, (p_1, m_1), \dots, (p_n, m_n)) := e(F) - \sum (1 - \frac{1}{m_i})$$

that by and large determines the geometry of M . For instance, M has spherical geometry iff $e^{orb} > 0$ and $H_1(M, \mathbb{Q}) = 0$ (see, for instance, [Sco83]).

Higher dimensional Seifert fibered manifolds were introduced and investigated in [OW75]. Roughly speaking, these are $(2n + 1)$ -manifolds L which admit a differentiable map $f : L \rightarrow X$ to a *complex* n -manifold X such that every fiber is a circle. Insisting that the base of a Seifert bundle be a complex manifold seems very artificial from the topological point of view, but a remarkable result a Kobayashi [Kob63] implies that for our purposes this is necessary. (See (12) for a precise statement.)

The natural setting seems to be to study Seifert bundles $f : L \rightarrow X$, where the base is a complex *locally cyclic orbifold*. That is, locally it looks like \mathbb{C}^n/G where G is a cyclic group acting linearly. There is a divisor $\cup D_i \subset X$ such that $L \rightarrow X$ is a circle bundle over $X \setminus \cup D_i$ and there are natural multiplicities m_i are assigned to the fibers over each D_i , see (14). We call $\Delta := \sum (1 - \frac{1}{m_i}) D_i$ the *branch divisor*, and we denote the base orbifold by (X, Δ) . The key invariant is the *orbifold Chern class*

$$c_1^{orb}(X, \Delta) := c_1(X) - \sum (1 - \frac{1}{m_i}) [D_i] \in H^2(X, \mathbb{Q}).$$

Another invariant, which enters into the final picture surprisingly little, is the Chern class of the Seifert bundle $c_1(L/X) \in H^2(X, \mathbb{Q})$, to be defined in (18).

Let (X, g) be a Riemannian manifold and $L \rightarrow X$ a circle bundle over X with a connection. At each point of L the connection decomposes the tangent space into a vertical and a horizontal piece. By choosing the standard metric on the circle fibers, we can lift the metric g to a metric g_L on L . With suitable care, this also works for Seifert bundles $f : L \rightarrow (X, \Delta)$.

Theorem 1. [Kob63, BG00, BGN03a] *Let (X, Δ) be a compact, complex orbifold and $f : L \rightarrow (X, \Delta)$ a Seifert bundle. Assume that $c_1(L/X)$ is a positive rational multiple of $c_1^{orb}(X, \Delta)$.*

Then, a positive Ricci curvature orbifold Kähler metric (resp. Kähler–Einstein metric) on (X, Δ) can be lifted to a positive Ricci curvature metric (resp. Einstein metric) on L .

Moreover, the lifted metric is also Sasakian.

It should be noted that a negative Ricci curvature Kähler–Einstein metric on (X, Δ) does not lift to a negative Ricci curvature Einstein metric on L in any natural way.

There are only a handful of cases when $L \rightarrow X$ is a circle bundle and the above theorem applies, but Boyer and Galicki [BG00, BG01] observed that many more cases appear when X is allowed to be an orbifold.

The method of Boyer and Galicki starts with a complex hypersurface $0 \in Y \subset \mathbb{C}^n$ with an isolated singularity at the origin which is invariant under a \mathbb{C}^* -action $(z_1, \dots, z_n) \mapsto (\lambda^{w_1} z_1, \dots, \lambda^{w_n} z_n)$. The intersection of Y with the unit sphere $L := Y \cap S^{2n-1}(1)$ is called the *link* of $0 \in Y$. L is a $(2n - 3)$ -dimensional real manifold with an S^1 -action. The quotient $X := L/S^1 \cong (Y \setminus \{0\})/\mathbb{C}^*$ is naturally a complex orbifold.

Methods of complex singularity theory allow one to identify L as a manifold, and this leads to a large class of new Einstein metrics on various spaces, including spheres and exotic spheres [BGK04].

The aim of this paper is to further generalize this construction to arbitrary Seifert bundles. The key advantage of this approach is that we can start with an orbifold (X, Δ) and construct Seifert bundles over (X, Δ) later. This provides substantially greater flexibility, allowing one to explore the natural scope of the theory. The construction of higher dimensional Seifert bundles was considered in [OW75] for X smooth; the general case is treated in [Kol04b]. The integral cohomology of Seifert bundles is rather subtle in general, but the 5-dimensional case is quite manageable.

The natural questions to be considered can be grouped around four problems:

Problems 2.

- (1) Describe all manifolds L which have a Seifert bundle structure $f : L \rightarrow (X, \Delta)$ over an orbifold with positive orbifold Chern class $c_1^{orb}(X, \Delta)$.
- (2) Given a manifold L , describe all Seifert bundle structures $f : L \rightarrow (X, \Delta)$ over an orbifold with $c_1^{orb}(X, \Delta) > 0$.
- (3) Construct positive Ricci curvature orbifold Kähler–Einstein metrics on orbifolds (X, Δ) .
- (4) Study the place of the resulting Einstein metrics in the theory of Einstein manifolds.

It seems rather artificial to separate the first two problems, but they are quite different in nature. I believe that in dimension five Problem (1) is doable with the present methods, whereas Problem (2) seems hopeless to me. The reason for this is connected with the complex geometry of the quotients $X = L/S^1$. These are *Del Pezzo* surfaces with quotient singularities, also called *log Del Pezzo* surfaces. While smooth Del Pezzo surfaces have been classified and understood for more than a century, log Del Pezzo surfaces occur in bewildering abundance and complexity; see, for instance, [Miy01, KM99, Sho00].

Nonetheless, as we see, Seifert bundles over log Del Pezzo surfaces tend to have simple topology. Thus many cases are excluded, and for some of the extreme cases one can get a complete description. Having a Seifert bundle with simple topology imposes only mild conditions on a log Del Pezzo surface, and I do not see how to get a good description. In all likelihood, the hardest is to describe all Seifert bundle structures on S^5 .

By Myers' theorem, the fundamental group of a compact manifold with positive Ricci curvature is finite. Therefore we concentrate on those cases when L is simply

connected. We see in (90) that for every simply connected manifold with such a Seifert bundle structure, the second Stiefel–Whitney class is zero. These 5-manifolds are well understood topologically:

Theorem 3. [Sma62] *Let L be a simply connected compact 5-manifold with vanishing second Stiefel–Whitney class. Then:*

- (1) L is uniquely determined by $H_2(L, \mathbb{Z})$.
- (2) The torsion subgroup of the second homology is of the form $\text{tors } H_2(L, \mathbb{Z}) \cong A + A$ for some finite Abelian group A .
- (3) For any finite Abelian group A and for any $n \geq 0$ there is a (unique) L with $H_2(L, \mathbb{Z}) \cong \mathbb{Z}^k + A + A$.

If $H_2(L, \mathbb{Z})$ is torsionfree of rank k then L is the connected sum of k copies of $S^2 \times S^3$. For any k , Einstein metrics on these were constructed in [BGN02, BGN03b, BG03, Kol04a]. By contrast, the first result of this paper shows that very few of the possible torsion subgroups do occur.

Theorem 4. *Let L be a compact 5-manifold such that $H_1(L, \mathbb{Z}) = 0$. Assume that L has a Seifert bundle structure $f : L \rightarrow (S, \Delta)$ with $c_1^{\text{orb}}(S, \Delta) > 0$. Then the torsion subgroup of the second homology, $\text{tors } H_2(L, \mathbb{Z})$, is one of the following:*

- (1) $(\mathbb{Z}/m)^2$ for any m ,
- (2) $(\mathbb{Z}/5)^4$ or $(\mathbb{Z}/4)^4$,
- (3) $(\mathbb{Z}/3)^4, (\mathbb{Z}/3)^6$ or $(\mathbb{Z}/3)^8$,
- (4) $(\mathbb{Z}/2)^{2n}$ for any n .

Conversely, all these cases do occur, even for manifolds with Einstein metrics.

One can be even more precise if $H_2(L, \mathbb{Z})$ is torsion, that is, when L is a *rational homology sphere*. The following characterization uses the notion of the *orbifold fundamental group* $\pi_1^{\text{orb}}(X, \Delta)$ and its abelianization $H_1^{\text{orb}}(X, \Delta)$, to be defined in (35).

Theorem 5. *There is a one-to-one correspondence between*

- (1) *Seifert bundle structures $f : L \rightarrow (S, \Delta)$ on 5-dimensional, compact rational homology spheres with $H_1(L, \mathbb{Z}) = 0$, and*
- (2) *compact, complex, 2-dimensional, locally cyclic orbifolds (S, Δ) with $H_2(S, \mathbb{Q}) = \mathbb{Q}$ and $H_1^{\text{orb}}(S, \Delta) = 0$.*

Under this correspondence, $\pi_1(L) = \pi_1^{\text{orb}}(S, \Delta)$.

There are many such orbifolds (S, Δ) where $c_1^{\text{orb}}(S, \Delta)$ is negative, but very few when $c_1^{\text{orb}}(S, \Delta)$ is positive. Three infinite series were found in [BG04a], giving Einstein metrics on rational homology spheres L with $H_2(L, \mathbb{Z}) = (\mathbb{Z}/m)^2$ for any m not divisible by 6. We give a complete classification of all cases when $H_2(L, \mathbb{Z})$ contains a torsion element of large enough order. Besides the above 3 series, there is only one more infinite series, but dozens, probably hundreds, of sporadic examples.

Theorem 6. *Let L be a 5-dimensional compact rational homology sphere which has a Seifert bundle structure $f : L \rightarrow (S, \Delta)$ with $c_1^{\text{orb}}(S, \Delta) > 0$. Assume that $H_1(L, \mathbb{Z}) = 0$ and $H_2(L, \mathbb{Z})$ contains a torsion element of order at least 12. Then:*

- (1) $H_2(L, \mathbb{Z}) = (\mathbb{Z}/m)^2$ for some m not divisible by 30,
- (2) L is simply connected,
- (3) *the number of Seifert bundle structures varies between 1 and 4, depending on $m \bmod 30$,*

- (4) *each Seifert bundle structure gives rise to a 2-parameter family of Einstein metrics, naturally parametrized by the moduli space of genus 1 curves.*

(See (84) for a detailed description of the orbifolds (S, Δ) .)

Remark 7. There are further examples with $H_2(L, \mathbb{Z}) = (\mathbb{Z}/m)^2$ for every $m \leq 11$, see (57, 58).

The rank of $H_2(L, \mathbb{Z})$ and its torsion subgroup are not independent of each other, but I have only fragmentary results describing their interaction.

Theorem 8. *Let L be a 5-dimensional compact manifold which has a Seifert bundle structure $f : L \rightarrow (S, \Delta)$ with $c_1^{orb}(S, \Delta) > 0$. Assume that $H_1(L, \mathbb{Z}) = 0$ and write $H_2(L, \mathbb{Z}) = \mathbb{Z}^n + A + A$.*

- (1) *If $H_2(L, \mathbb{Z}) = \mathbb{Z}^n + (\mathbb{Z}/m)^2$ for some $m \geq 12$ then*
 - (a) *L is simply connected and $n \leq 8$,*
 - (b) *there are 93 such cases for every $m \geq 12$, (see (70) for the complete list)*
 - (c) *each Seifert bundle structure yields an at least 2-parameter family of Einstein metrics.*
- (2) *If $H_2(L, \mathbb{Z}) = \mathbb{Z}^n + (\mathbb{Z}/5)^4$ then*
 - (a) *L is a simply connected rational homology sphere,*
 - (b) *L and the Seifert bundle structure are unique, and*
 - (c) *L admits a 4-dimensional family of Einstein metrics, naturally parametrized by the moduli space of genus 2 curves.*

9 (Kähler–Einstein metrics). In Section 7 we construct positive Ricci curvature Kähler–Einstein metrics on certain 2-dimensional orbifolds. While these examples are mostly new, the method is the same as in [JK01], relying on earlier works of [Nad90, DK01]. Thus nothing essentially new is added to Problem (3).

10 (Sasakian manifolds). While I prefer to think of Seifert bundles as a topological object L associated to an algebro-geometric object (X, Δ) , they have a natural place within the framework of Sasakian geometry. (See [BG04b] for a recent survey paper.)

Roughly speaking, a *quasi-regular Sasakian manifold* is a Seifert bundle L over a Kähler orbifold (X, Δ) plus a metric on L which optimally matches the orbifold Kähler metric on (X, Δ) .

In the language of Sasakian geometry, the main results of this paper are the following:

- (1) Corollary (81) gives topological restrictions for a 5-dimensional rational homology sphere to admit a quasi-regular Sasakian structure.
- (2) Theorem (4) shows that most 5-manifolds do not admit a positive quasi-regular Sasakian structure.
- (3) Theorems (6) and (8) classify all positive quasi-regular Sasakian structures on certain 5-manifolds.
- (4) We also get new examples of quasi-regular Sasakian–Einstein metrics, though the examples discovered by Boyer and Galicki already cover almost all cases allowed by Theorems (4) and (6).

11 (Description of the sections). Basic results on Seifert bundles and on the Kobayashi construction are recalled in Section 1. In the most general case, we are led to study

Seifert bundles over orbifolds which locally look like the quotient of \mathbb{C}^n by a cyclic group. These are studied in Section 2.

Our main aim is to understand 5-dimensional Seifert bundles where the base is a log Del Pezzo surface (53). These are rational surfaces with quotient singularities. In Section 3 we see how to compute the topological (co)homology of these surfaces in terms of their algebraic geometry. A key point is to compute everything with \mathbb{Z} -coefficient.

The cohomology groups of a Seifert bundle $f : L \rightarrow S$ are computed by a Leray spectral sequence, and we study it in Section 4. The spectral sequence degenerates at E_3 and we get a pretty complete description of the E_2 -term. The differential $E_2^{0,1} \rightarrow E_2^{2,0}$ is identified with a first Chern class. A key observation is that torsion in $H_2(L, \mathbb{Z})$ comes from curves $C \subset S$ of genus at least 1 such that every fiber of f above C is multiple.

Log Del Pezzo surfaces have been the objects of intense investigation (see [Miy01, KM99, Sho00]). While smooth Del Pezzo surfaces form a well understood and easy to describe class, log Del Pezzo surfaces are rather numerous. Nonetheless, it is quite rare that a Seifert bundle can have multiple fibers over a curve of genus at least 1, and many of these are classified in Section 5.

Del Pezzo surfaces with cyclic Du Val singularities and simply connected smooth part are listed in Section 6.

Section 7 establishes the existence of Kähler–Einstein metrics on most of the surfaces considered previously.

A characterization of Seifert bundle structures on rational homology spheres is given in Section 8 in terms of algebraic geometry. There are probably too many cases for a meaningful classification. Here we also collect the details to get proofs of the main theorems.

There is a close relationship between manifolds with Seifert bundles and links of 3-dimensional log terminal singularities. Some of the resulting open questions are mentioned in Section 9.

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1. EINSTEIN METRICS ON SEIFERT BUNDLES

Let (X, g) be a Riemannian manifold and $L \rightarrow X$ a circle bundle over X with a connection D . At each point of L the connection decomposes the tangent space into a vertical and a horizontal piece. By choosing the standard metric on the circle fibers, we can lift the metric g to a metric g_D on L . Kobayashi [Kob63] asked: If (X, g) is an Einstein manifold, when is (L, g_D) Einstein?

It turns out that this happens only rarely.

Theorem 12. [Kob63] *Notation and assumptions as above. Then (L, g_D) is an Einstein manifold iff and only if one of the following two conditions are satisfied:*

- (1) (X, g) is Ricci flat and D is flat.
- (2) X is a complex manifold, g is the real part of a Kähler–Einstein metric ω , $\text{Ricci}(\omega) > 0$ and the curvature of D is a positive multiple of $\text{Ricci}(\omega)$.

The first case is essentially trivial, and here we are interested in the second case.

In any given dimension, there are only finitely many deformation types of compact complex manifolds with positive Ricci curvature (cf. [Nad91, Cam91, KMM92]), so (12.2) gives only a handful of new Einstein manifolds.

If X is a compact complex manifold with positive Ricci curvature then $H^i(X, \mathcal{O}_X) = 0$ for every $i > 0$, in particular, every topological circle bundle over X can be written uniquely as the unit circle bundle of a holomorphic \mathbb{C}^* -bundle $f : Y \rightarrow X$. Thus the second alternative of (12) can be formulated completely in terms of complex geometry. This is the setting where one can generalize the Kobayashi construction.

Definition 13. Let X be a normal complex space. A *Seifert \mathbb{C}^* -bundle* over X is a normal complex space Y together with a morphism $f : Y \rightarrow X$ and a \mathbb{C}^* -action on Y satisfying the following two conditions.

- (1) f is Stein (that is, the preimage of any open Stein subset of X is Stein) and \mathbb{C}^* -equivariant (with respect to the trivial action on X).
- (2) For every $x \in X$, the \mathbb{C}^* -action on the reduced fiber $Y_x := \text{red } f^{-1}(x)$, $\mathbb{C}^* \times Y_x \rightarrow Y_x$ is \mathbb{C}^* -equivariantly biholomorphic to the natural \mathbb{C}^* -action on \mathbb{C}^*/μ_m for some $m = m(x, Y/X)$, where $\mu_m \subset \mathbb{C}^*$ denotes the group of m th roots of unity.

The number $m(x, Y/X)$ is called the *multiplicity* of the Seifert fiber over x .

One can always assume that the \mathbb{C}^* -action is effective, that is, $m(x, Y/X) = 1$ for general $x \in X$.

Note that even if Y is smooth, X can have quotient singularities.

A classification of Seifert \mathbb{C}^* -bundles for X a smooth manifold with $H_1(X, \mathbb{Z})$ torsion free is given in [OW75]. Many other cases are described in [Dol75, Pin77, Dem88, FZ03]. The general case is discussed in detail in [Kol04b]. We recall the relevant facts below.

14 (Description of Seifert \mathbb{C}^* -bundles). Let $f : Y \rightarrow X$ be a Seifert \mathbb{C}^* -bundle. The set of points $\{x \in X : m(x, Y/X) > 1\}$ is a closed analytic subset of X . It can be written as the union of Weil divisors $\cup D_i$ and of a subset of codimension at least 2 contained in $\text{Sing } X$. The latter will not be relevant to us. The multiplicity $m(x, Y/X)$ is constant on a dense open subset of each D_i , this common value is called the multiplicity of the Seifert \mathbb{C}^* -bundle over D_i ; denote it by $m_i := m(D_i)$. We call the \mathbb{Q} -divisor $\Delta := \sum (1 - \frac{1}{m_i})D_i$ the *branch divisor* of $f : Y \rightarrow X$. We frequently write $f : Y \rightarrow (X, \Delta)$ to indicate the branch divisor.

Theorem 15. [Kol04b, Thm.7] *Let X be a normal complex space with quotient singularities and $\Delta := \sum (1 - \frac{1}{m_i})D_i$ a \mathbb{Q} -divisor. There is a one-to-one correspondence between Seifert \mathbb{C}^* -bundles $f : Y \rightarrow (X, \Delta)$ and the following data:*

- (1) *For each D_i an integer $0 \leq b_i < m_i$, relatively prime to m_i , and*
- (2) *a linear equivalence class of Weil divisors $B \in \text{Weil}(X)$. (In all the cases that we consider in this paper, $\text{Weil}(X) \cong H_2(X, \mathbb{Z})$.)*

16 (Algebraic construction). [Kol04b, Thm.7] Given the above data, set

$$\begin{aligned} S(B, \sum \frac{b_i}{m_i} D_i) &:= \sum_{j \in \mathbb{Z}} \mathcal{O}_X(jB + \sum_i \lfloor jb_i/m_i \rfloor D_i) \quad \text{and} \\ Y(B, \sum \frac{b_i}{m_i} D_i) &:= \text{Spec}_X S(B, \sum \frac{b_i}{m_i} D_i), \end{aligned}$$

where $\lfloor x \rfloor$ denotes the round down or integral part of a real number.

There is a natural G_m -action on $S(L, \sum \frac{b_i}{m_i} D_i)$ where the j th summand is the λ^j eigensubsheaf. This defines a \mathbb{C}^* -action on $Y(L, \sum \frac{b_i}{m_i} D_i)$.

17 (Topological construction). [OW75, 3.8] Set $M(\Delta) = \text{lcm}(m_i)$. If X is smooth then $Y \setminus f^{-1}(\text{Supp } \Delta)$ can be written as an $M(\Delta)$ -sheeted covering space of the \mathbb{C}^* -bundle over $X \setminus \text{Supp } \Delta$ with Chern class $M(\Delta) \cdot ([B] + \sum \frac{b_i}{m_i} [D_i])$. The precise description needs the calculation of $H_1(Y \setminus f^{-1}(\text{Supp } \Delta), \mathbb{Z})$ given in [OW75, 2.1]. (Note that $M(\Delta) \cdot ([B] + \sum \frac{b_i}{m_i} [D_i])$ can be viewed as a well defined integral cohomology class, since $m_i | M(\Delta)$. Dealing with torsion in $H^2(X, \mathbb{Z})$ is, however, delicate, and [OW75] assumes that there is no torsion.)

If X is singular, then one first proves that $f : Y \rightarrow (X, \Delta)$ is uniquely determined by its restriction to the smooth locus of X and then the previous method applies.

Definition 18. The *Chern class* of $f : Y \rightarrow (X, \Delta)$ is defined as

$$c_1(Y/X) := [B] + \sum \frac{b_i}{m_i} [D_i] \in H^2(X, \mathbb{Q}).$$

Definition 19 (Orbifolds). An *orbifold* is a normal, compact, complex space X locally given by charts written as quotients of smooth coordinate charts. That is, X can be covered by open charts $X = \cup U_i$ and for each U_i there is a smooth complex space V_i and a finite group G_i acting on V_i such that U_i is biholomorphic to the quotient space V_i/G_i . The quotient maps are denoted by $\phi_i : V_i \rightarrow U_i$.

The compatibility condition between the charts that one needs to assume is that there are global divisors $D_j \subset X$ and ramification indices m_j such that $D_{ij} = U_i \cap D_j$ and $m_{ij} = m_j$ (after suitable re-indexing).

It is convenient to codify these data by a single \mathbb{Q} -divisor, called the *branch divisor* of the orbifold, $\Delta := \sum (1 - \frac{1}{m_j}) D_j$.

The *orbifold canonical class* is $K_X + \sum (1 - \frac{1}{m_i}) D_i$. This is the negative of the *orbifold Chern class*

$$c_1^{\text{orb}}(X, \Delta) := c_1(X) - \sum (1 - \frac{1}{m_i}) [D_i] \in H^2(X, \mathbb{Q}).$$

Example 20. Let $f : Y \rightarrow (X, \Delta)$ be a Seifert \mathbb{C}^* -bundle with Y smooth. For $x \in X$ pick any $y \in f^{-1}(x)$ and a μ_m -invariant smooth hypersurface $V_x \subset Y$ transversal to $\text{red } f^{-1}(x)$ for $m = m(x, Y/X)$. Then $\{\phi_x : V_x \rightarrow U_x := V_x/\mu_m\}$ gives an orbifold structure on X . The orbifold branch divisor coincides with the branch divisor of the Seifert bundle defined in (14).

Note that the orbifolds coming from a smooth Seifert bundle have the additional property that each U_x is a quotient by a cyclic group μ_m . Such an orbifold is called *locally cyclic*.

We usually identify (X, Δ) with this orbifold structure.

Definition 21 (Metrics on orbifolds). A *Hermitian metric* h on the orbifold (X, Δ) is a Hermitian metric h on $X \setminus (\text{Sing } X \cup \text{Supp } \Delta)$ such that for every chart $\phi_i : V_i \rightarrow U_i$ the pull back $\phi_i^* h$ extends to a Hermitian metric on V_i . One can now talk about curvature, Kähler metrics, Kähler–Einstein metrics on orbifolds.

Definition 22. As a real Lie group, $\mathbb{C}^* \cong S^1 \times \mathbb{R}$, thus every Seifert \mathbb{C}^* -bundle contains a real hypersurface $L \subset Y$ with an S^1 -action. We call $f : L \rightarrow (X, \Delta)$ a *Seifert S^1 -bundle* or simply a *Seifert bundle*. (In dimension 3, these are the original Seifert bundles.)

Y retracts to L , thus they have isomorphic homology and homotopy groups.

Theorem 23. [Kob63, BG00] Let $f : Y \rightarrow (X, \Delta = \sum (1 - \frac{1}{m_i}) D_i)$ be a Seifert \mathbb{C}^* -bundle and $f : L \rightarrow (X, \Delta)$ the corresponding Seifert S^1 -bundle. Assume that L

is a manifold. Then L admits an S^1 -invariant Einstein metric with positive Ricci curvature if and only if the following hold.

- (1) The orbifold canonical class $K_X + \Delta$ is anti ample and there is an orbifold Kähler–Einstein metric on (X, Δ) .
- (2) The Chern class $c_1(Y/X)$ is a negative multiple of $K_X + \Delta$. \square

From the point of view of algebraic geometry, the most useful property is the first part of (23.1). This class of orbifolds have their own name.

Definition 24. An orbifold (X, Δ) is called *Fano* or *log Fano* if the orbifold canonical class $K_X + \Delta$ is anti ample.

2. SMOOTH SEIFERT BUNDLES

25 (Locally cyclic orbifolds). Let $f : Y \rightarrow (X, \Delta)$ be a Seifert \mathbb{C}^* -bundle, $\dim X = n$. Pick a point $x \in X$ and assume that Y is smooth along $f^{-1}(x)$. As we saw in (20), (X, Δ) is then a locally cyclic orbifold near x . By diagonalizing the cyclic group action, we see that locally $x \in X$ is biholomorphic to D^n/μ_m for $m = m(x, Y/X)$ where $D^n \subset \mathbb{C}^n$ is the n -dimensional polydisc and the irreducible components of Δ passing through x are the quotients of (some of) the coordinate hyperplanes in D^n .

A straightforward explicit computation (cf. [Kol04b, 22–25]) shows that $m = r \cdot m_1 \cdots m_n$ where

- (1) The m_1, \dots, m_n are the multiplicities of the irreducible components of Δ passing through x . (We add the necessary number of 1-s if there are fewer than n such components.)
- (2) The m_1, \dots, m_n are pairwise relatively prime.
- (3) The quotient $\bar{D}^n := D^n/\mu_{m_1 \cdots m_n}$ is smooth, $D^n/\mu_m \cong \bar{D}^n/\mu_r$ and the μ_r -action is fixed point free outside a codimension 2 subset.
- (4) Thus \mathbb{Z}/r is the local fundamental group of $X \setminus \text{Sing } X$ at x , hence $m(x, Y/X)$ depends only on (X, Δ) and not on Y .

Definition 26. Let (X, Δ) be an orbifold given by the charts $\phi_i : V_i \rightarrow U_i$. A divisor $D \subset X$ is called *orbisooth* if the set theoretic preimages $\phi_i^{-1}(D) \subset V_i$ are all smooth.

Thus D itself is an orbifold with the induced orbifold structure.

If $\dim X = 2$ then D is a curve and orbisooth implies smooth. However, not all smooth curves in X are orbisooth.

For instance, act on \mathbb{C}^2 by μ_5 as $(x, y) \mapsto (\epsilon^2 x, \epsilon^3 y)$. Then $(x^3 - y^2 = 0)/\mu_5 \subset \mathbb{C}^2/\mu_5$ is smooth but not orbisooth.

Definition 27. Let $(X, \Delta = \sum(1 - \frac{1}{m_i})D_i)$ be a locally cyclic orbifold. Such an orbifold behaves very much like a manifold if we use \mathbb{Q} -coefficients, but torsion questions become quite delicate when working with integral (co)homology. We need several ways to measure contributions of the orbifold points.

As above, for every $x \in X$ we can write the orbifold structure of (X, Δ) in a suitable neighborhood as $D^n/\mu_{m(x)}$ and the orbifold structure of (X, \emptyset) as $\bar{D}^n/\mu_{r(x)}$.

- (1) Set $M(x, \Delta) := \text{lcm}(m_i : x \in D_i)$ and $M(\Delta) := \text{lcm}(m(x, \Delta) : x \in X) = \text{lcm}(m_i)$.
- (2) Set $M(x, X) = r(x)$ and $M(X) := \text{lcm}(M(x, X)) : x \in X$

- (3) Set $M(x, X, \Delta) = m(x)$ and $M(X, \Delta) := \text{lcm}(M(x, X, \Delta) : x \in X) = \text{lcm}(m(x) : x \in X)$.

As noted in (25), $M(x, X, \Delta) = M(x, \Delta) \cdot M(x, X)$ and $M(X, \Delta) | M(\Delta) \cdot M(X)$ but the latter can be different.

Definition 28. Let μ_r act on D^n such that the action is fixed point free outside a codimension 2 subset. Then $\text{Weil}(D^n/\mu_r) \cong \mathbb{Z}/r$, noncanonically (cf. [Kol04b, 24]). This is called the *local class group* of X at x , denoted by $\text{Weil}(x, X)$. One can also identify this group as the second cohomology of the smooth part of the quotient D^n/μ_r . Thus if X is an orbifold, for every singular point of X we obtain a map $R_x : \text{Weil}(X) \rightarrow \text{Weil}(x, X)$ which is nonzero on a Weil divisor A iff A is not Cartier at x . Thus

$$\text{Pic}(X) = \bigcap_{x \in X} \ker R_x \subset \text{Weil}(X).$$

Topologically, we can see this as

$$H^2(X, \mathbb{Z}) = \bigcap_x \ker[H_{2n-2}(X, \mathbb{Z}) \rightarrow \text{Weil}(x, X)],$$

where we identify $H^2(X, \mathbb{Z})$ with its image in $H_{2n-2}(X, \mathbb{Z})$ by capping with the fundamental class.

In particular, we can view multiplication by $M(X)$ as a map

$$M(X) : \text{Weil}(X) \rightarrow \text{Pic}(X) \quad \text{or} \quad M(X) : H_{2n-2}(X, \mathbb{Z}) \rightarrow H^2(X, \mathbb{Z}).$$

Definition 29. Let $f : Y \rightarrow (X, \Delta)$ be a Seifert \mathbb{C}^* -bundle over the orbifold (X, Δ) . Its Chern class

$$c_1(Y/X) = [B] + \sum \frac{b_i}{m_i} [D_i].$$

is an element of $H^2(X, \mathbb{Q})$, but certain multiples of it can be viewed as well defined integral classes, even in the presence of torsion. We need three versions of this:

- (1) $M(\Delta) \cdot c_1(Y/X) := M(\Delta) \cdot [B] + \sum \frac{M(\Delta)}{m_i} b_i \cdot [D_i]$ is a well defined element of $\text{Weil}(X)$ or of $H_{2n-2}(X, \mathbb{Z})$.
- (2) $M(x, \Delta) \cdot c_1(Y/X) := M(x, \Delta) M \cdot [B] + \sum_{i: x \in D_i} \frac{M(x, \Delta)}{m_i} b_i \cdot [D_i]$ is a well defined element of $\text{Weil}(x, X)$.
- (3) $M(X, \Delta) \cdot c_1(Y/X) := M(X, \Delta) \cdot [B] + \sum \frac{M(X, \Delta)}{m_i} b_i \cdot [D_i]$ is a well defined element of $\text{Pic}(X)$ or of $H^2(X, \mathbb{Z})$.

In general, a Seifert \mathbb{C}^* -bundle over an orbifold is singular, but the smooth ones are easy to determine:

Proposition 30 (Smoothness criterion). [Kol04b, 29] *Let $Y := Y(B, \sum \frac{b_i}{m_i} D_i)$ be a Seifert bundle over the orbifold $(X, \sum (1 - \frac{1}{m_i}) D_i)$. Then Y is smooth along $f^{-1}(x)$ iff $M(x, \Delta) \cdot c_1(Y/X)$ is a generator of the local class group $\text{Weil}(x, X)$. \square*

3. THE TOPOLOGY OF SINGULAR SURFACES

Let $L \rightarrow (S, \Delta)$ be a Seifert bundle over a compact complex surface. As we saw in (23), the Kobayashi construction yields an Einstein metric on L only if $-(K_S + \Delta)$ is ample. In particular, S is always a projective algebraic surface. If L is smooth then S has only quotient singularities by (20). Such surface is always rational (this is an easy special case of [Sak84] or of [MM86]).

In practice, one can understand the algebraic curves and their intersection theory on any (singular) rational surface. The aim of this section is to describe their topology, especially various (co)homology groups, in terms of algebraic curves.

Let S be a normal compact surface. Let S^0 denote the smooth locus and $P_i \in S$ the set of singular points. Topologically, near any singular point S is a cone C_i over a 3-manifold M_i called the *link*. While we are mainly interested in surfaces with cyclic quotient singularities, it is equally easy to work with arbitrary rational singularities.

Definition 31. Let $0 \in F$ be a normal surface singularity and $g : F' \rightarrow F$ a resolution. The singularity $0 \in F$ is *rational* if $R^1g_*\mathcal{O}_{F'} = 0$. This is independent of the resolution chosen.

If $0 \in F$ is rational then $g^{-1}(0) \subset F'$ is a tree of smooth rational curves. This implies that $R^1g_*\mathbb{Z} = 0$ and $H^1(M, \mathbb{Z}) = 0$ where M is the link of $0 \in F$. See [Mum61] for these and more information on the topology of surface singularities.

Proposition 32. *Let S be a normal, compact surface with rational singularities P_i and links M_i . Assume that $H^1(S, \mathbb{Z}) = 0$. Let $S^0 \subset S$ be the smooth locus. Then*

- (1) $H^3(S, \mathbb{Z}) = H_1(S^0, \mathbb{Z})$ is torsion.
- (2) $H_2(S^0, \mathbb{Z}) \cong H^2(S, \mathbb{Z})$ is torsion free.
- (3) $H^2(S^0, \mathbb{Z}) \cong H_2(S, \mathbb{Z})$.
- (4) We can write $H_2(S, \mathbb{Z})$ as the direct sum of a free group $H_2(S, \mathbb{Z})_{tf}$ and a torsion group isomorphic to $H_1(S^0, \mathbb{Z})$.
- (5) There is an exact sequence

$$0 \rightarrow H^2(S, \mathbb{Z}) \rightarrow H_2(S, \mathbb{Z})_{tf} + H_1(S^0, \mathbb{Z}) \rightarrow \sum_i H^2(M_i, \mathbb{Z}) \rightarrow H_1(S^0, \mathbb{Z}) \rightarrow 0.$$

- (6) $H^3(S^0, \mathbb{Z}) \cong \mathbb{Z}^s$ where s is the number of singular points.

Proof. Set $S^* := S \setminus \cup_j \text{interior}(C_j)$. The inclusion $S^* \subset S^0$ is a homotopy equivalence.

The long exact cohomology sequence of the pair (S, S^*) gives

$$H^i(S, S^*, \mathbb{Z}) \rightarrow H^i(S, \mathbb{Z}) \rightarrow H^i(S^*, \mathbb{Z}) \rightarrow H^{i+1}(S, S^*, \mathbb{Z}) \rightarrow \dots$$

By excision, $H^{i+1}(S, S^*, \mathbb{Z})$ is the direct sum of the local terms $H^{i+1}(C_j, M_j, \mathbb{Z})$. C_j is contractible, so $H^i(C_j, \mathbb{Z}) = 0$ for $i \geq 1$, thus $H^{i+1}(C_j, M_j, \mathbb{Z}) = H^i(M_j, \mathbb{Z})$ for $i \geq 1$.

For any rational surface singularity $H^1(M_i, \mathbb{Z}) = 0$ and $H^2(M_i, \mathbb{Z})$ is torsion, thus we get an exact sequence

$$0 \rightarrow H^2(S, \mathbb{Z}) \rightarrow H^2(S^*, \mathbb{Z}) \rightarrow \sum_j H^2(M_j, \mathbb{Z}) \rightarrow H^3(S, \mathbb{Z}) \rightarrow H^3(S^*, \mathbb{Z}).$$

Assume next that $H_1(S, \mathbb{Z}) = 0$. Let $g : S' \rightarrow S$ be a resolution of singularities. Since S has rational singularities, the fibers of g are all simply connected, thus $H_1(S', \mathbb{Z}) = 0$ and hence $H^3(S', \mathbb{Z}) = 0$. Therefore $H^3(S, \mathbb{Z}) \rightarrow H^3(S^*, \mathbb{Z})$ is the zero map since it factors as $H^3(S, \mathbb{Z}) \rightarrow H^3(S', \mathbb{Z}) \rightarrow H^3(S^*, \mathbb{Z})$. In particular, $H^3(S^*, \mathbb{Z}) \cong \sum_j H^3(M_j, \mathbb{Z}) \cong \mathbb{Z}^s$ if we have s singular points.

Alexander duality for $S' \supset \cup_j f^{-1}(C_j)$ gives isomorphisms

$$H^i(S^*, \mathbb{Z}) \cong H_i(S', \cup_j f^{-1}(C_j), \mathbb{Z}) = H_i(S', \cup_j f^{-1}(P_j), \mathbb{Z}).$$

By excision, $H_i(S', \cup_j f^{-1}(C_j), \mathbb{Z}) = H_i(S, \cup_j C_j, \mathbb{Z})$ which equals $H_i(S, \mathbb{Z})$ for $i \geq 2$. Finally, by the universal coefficient theorem and by Alexander duality,

$$\begin{aligned} \text{tors } H_1(S^*, \mathbb{Z}) &\cong \text{tors } H^2(S^*, \mathbb{Z}) \cong \text{tors } H_2(S', \cup_j f^{-1}(C_j), \mathbb{Z}) \\ &= \text{tors } H_2(S, \mathbb{Z}) = H^3(S, \mathbb{Z}). \end{aligned}$$

This completes the proof. \square

Note 33. The appearance of $H_1(S^0, \mathbb{Z})$ in (5) at two places puts severe restrictions on the group $H_1(S^0, \mathbb{Z})$, especially since $H^2(S, \mathbb{Z})$ is torsion free. Indeed, this implies that the sequence

$$0 \rightarrow (H_2(S, \mathbb{Z})_{tf} / H^2(S, \mathbb{Z})) + H_1(S^0, \mathbb{Z}) \rightarrow \sum_i H^2(M_i, \mathbb{Z}) \rightarrow H_1(S^0, \mathbb{Z}) \rightarrow 0$$

is exact. Thus if $\prod_i |H^2(M_i, \mathbb{Z})|$ is square free, we can conclude right away that $H_1(S^0, \mathbb{Z}) = 0$.

Corollary 34. *Let S be a normal, compact surface with rational singularities P_i with links M_i . Assume that $H_1(S, \mathbb{Z}) = 0$. The following are equivalent*

- (1) $H_1(S^0, \mathbb{Z}) = 0$.
- (2) $|H_2(S, \mathbb{Z}) / H^2(S, \mathbb{Z})| = \prod_i |H^2(M_i, \mathbb{Z})|$.
- (3) *The determinant of the intersection matrix on $H^2(S, \mathbb{Z})$ is $\prod_i |H^2(M_i, \mathbb{Z})|$.*

Proof. $H_1(S^0, \mathbb{Z}) = 0$ iff $H^3(S, \mathbb{Z}) = 0$ which is equivalent to (2) by the sequence (32.6). Since the pairing $H_2(S, \mathbb{Z})_{tf} \times H^2(S, \mathbb{Z}) \rightarrow \mathbb{Z}$ is perfect, $|H_2(S, \mathbb{Z}) / H^2(S, \mathbb{Z})|$ is the same as the determinant of the intersection matrix on $H^2(S, \mathbb{Z})$. \square

Definition 35. [Thu78] For an orbifold (X, Δ) the *orbifold fundamental group* $\pi_1^{orb}(X, \Delta = \sum_i (1 - \frac{1}{m_i}) D_i)$ is the fundamental group of $X \setminus (\text{Sing } X \cup \text{Supp } \Delta)$ modulo the relations: if γ is any small loop around D_i at a smooth point then $\gamma^{m_i} = 1$.

The abelianization of $\pi_1^{orb}(X, \Delta)$ is denoted by $H_1^{orb}(X, \Delta)$, called the *abelian orbifold fundamental group*.

Proposition 36. *Let S be a normal, projective surface with rational singularities P_i with links M_i . Then $H_1^{orb}(S, \sum_j (1 - \frac{1}{m_j}) D_j) = 0$ iff*

- (1) $H_1(S^0, \mathbb{Z}) = 0$, and
- (2) *The map $H^2(S, \mathbb{Z}) \rightarrow \sum_j \mathbb{Z}/m_j$ given by $L \mapsto (L \cdot D_j) \pmod{m_j}$ is surjective.*

Proof. By [OW75, 4.6], if $H_1(S^0, \mathbb{Z}) = 0$ then $H_1^{orb}(S, \Delta)$ is given by generators g_1, \dots, g_n and relations

- (3) $m_j g_j = 0$ for $j = 1, \dots, n$, and
- (4) $\sum g_j ([D_j] \cap \eta) = 0$ for every $\eta \in H_2(X, \mathbb{Z})$.

Thus there is an exact sequence

$$H_2(S^0, \mathbb{Z}) \xrightarrow{\sigma} \sum_j \mathbb{Z}/m_j \rightarrow H_1^{orb}(S, \Delta) \rightarrow 0$$

where $\sigma(\eta) = \sum ([D_j] \cap \eta) g_j$.

By (32.2), $H_2(S^0, \mathbb{Z})$ is isomorphic to $H^2(S, \mathbb{Z})$ and the map σ is identified with taking the intersection number with each D_i (modulo m_i). \square

Remark 37. A very easy to apply special case of (36) is the following:

If we can write

$$\sum_{i: p|m_i} a_i D_i \sim p \cdot (\text{integral Weil divisor})$$

and not all the a_i are divisible by p , then $H_1^{orb}(S, \Delta) \neq 0$. Indeed, for any $L \in H^2(S, \mathbb{Z})$ this would give

$$\sum_{i: p|m_i} a_i(L \cdot D_i) \equiv 0 \pmod{p}.$$

We now start to connect the previous results with the algebraic geometry of the surface S .

Proposition 38. *Let S be a normal compact surface with rational singularities such that $H^1(S, \mathcal{O}_S) = 0$. Let $g : S' \rightarrow S$ be a resolution of singularities. Then*

- (1) *There are injections with torsion free cokernels*

$$\text{Weil}(S) \hookrightarrow H_2(S, \mathbb{Z}) \quad \text{and} \quad \text{Pic}(S) \hookrightarrow H^2(S, \mathbb{Z}).$$

- (2) $H_2(S, \mathbb{Z}) / \text{Weil}(S) \cong H_2(S', \mathbb{Z}) / \text{Weil}(S')$.

- (3) *If $H^2(S, \mathcal{O}_S) = 0$ then both of the injections are isomorphisms.*

Proof. The long exact cohomology sequence of the exponential sequence

$$0 \rightarrow \mathbb{Z} \xrightarrow{2\pi i} \mathcal{O}_S \xrightarrow{\exp} \mathcal{O}_S^* \rightarrow 1$$

gives $\text{Pic}(S) \hookrightarrow H^2(S, \mathbb{Z})$ if $H^1(S, \mathcal{O}_S) = 0$ and this is an isomorphism if $H^2(S, \mathcal{O}_S) = 0$ also holds. The cokernel is torsion free since $H^2(S, \mathcal{O}_S)$ is torsion free.

Set $A := \text{Sing } S$ and $A' := f^{-1}(A)$. The homology sequences of the pairs (S', A') and (S, A) give a commutative diagram

$$\begin{array}{ccccccc} H_2(A', \mathbb{Z}) & \rightarrow & H_2(S', \mathbb{Z}) & \rightarrow & H_2(S', A', \mathbb{Z}) & \rightarrow & H_1(A', \mathbb{Z}) \\ \downarrow 0 & & \downarrow f_* & & \downarrow \cong & & \downarrow 0 \\ H_2(A, \mathbb{Z}) & \rightarrow & H_2(S, \mathbb{Z}) & \rightarrow & H_2(S, A, \mathbb{Z}) & \rightarrow & H_1(A, \mathbb{Z}) \end{array}$$

The horizontal maps at the end are 0 since $\dim A = 0$. $H_1(A', \mathbb{Z}) = 0$ since the resolution graph of a rational surface singularity is a tree of rational curves. This implies that $H_2(S, \mathbb{Z}) \cong H_2(S', \mathbb{Z}) / H_2(A', \mathbb{Z})$. $H_2(S', \mathbb{Z}) / \text{Weil}(S') \cong H^2(S', \mathbb{Z}) / \text{Pic}(S')$ is torsion free and $\text{im}[H_2(A', \mathbb{Z}) \rightarrow H_2(S', \mathbb{Z})] \subset \text{Weil}(S')$. Thus

$$H_2(S, \mathbb{Z}) / \text{Weil}(S) \cong H_2(S', \mathbb{Z}) / \text{Weil}(S')$$

is torsion free and it is zero iff $H^2(S, \mathcal{O}_S) = 0$. □

Remark 39. In general the natural map $H^2(S, \mathbb{Z}) / \text{Pic}(S) \rightarrow H^2(S', \mathbb{Z}) / \text{Pic}(S')$ is not an isomorphism. As an example, let $S' \subset \mathbb{P}^3$ be a general quartic surface containing a plane conic C . $C \subset S'$ is a -2 -curve and we contract it to get S . Then $H^2(S, \mathbb{Z}) = \ker[H^2(S', \mathbb{Z}) \rightarrow H^2(C, \mathbb{Z}) \cong \mathbb{Z}]$ and $\text{Pic}(S) = \ker[\text{Pic}(S') \rightarrow \text{Pic}(C) \cong \mathbb{Z}]$. The first of these is surjective while the image of the second is $2\mathbb{Z}$.

Proposition 40. *Let S be a normal compact surface with rational singularities.*

- (1) *If $H^1(S, \mathbb{Q}) = 0$ then $H^1(S, \mathcal{O}_S) = 0$.*
(2) *If, in addition, $H^2(S, \mathbb{Q}) \cong \mathbb{Q}$ then $H^2(S, \mathcal{O}_S) = 0$.*

Proof. Let $f : S' \rightarrow S$ be a resolution, then $R^1 f_* \mathbb{Z} = 0$ and $R^1 f_* \mathcal{O}_{S'} = 0$ since S has rational singularities. If $H^1(S, \mathbb{Q}) = 0$, the first implies that $H^1(S', \mathbb{Q}) = 0$ and so $H^1(S', \mathcal{O}_S) = 0$ by Hodge theory (cf. [GH78, Sec.0.7]). This gives $H^1(S, \mathcal{O}_S) = 0$.

If $H_2(S, \mathbb{Q}) \cong \mathbb{Q}$ then it is generated by algebraic cycles. By (38) this implies that $H_2(S', \mathbb{Q}) \cong \mathbb{Q}$ is generated by algebraic cycles, hence $H^2(S', \mathcal{O}_{S'}) = 0$ which gives that $H^2(S, \mathcal{O}_S) = 0$. □

41 (Surfaces with $H^1(F, \mathcal{O}_F) = H^2(F, \mathcal{O}_F) = 0$). By classification theory, smooth projective surfaces with $H^1(F, \mathcal{O}_F) = H^2(F, \mathcal{O}_F) = 0$ fall in three groups. (See, for instance, [BHPVdV04, Chap.VI] for a comprehensive treatment.)

- (1) F is a rational surface,
- (2) F is an Enriques surface [BHPVdV04, VIII.15–21], or
- (3) F is of general type with $q = p_g = 0$ [BHPVdV04, VII.10].

We are specially interested in cases when F is obtained as a resolution of a surface S with rational singularities and $H^2(S, \mathbb{Q}) \cong \mathbb{Q}$. There are many such examples where F is rational but probably very few examples where F is Enriques or of general type.

A nice class of examples is given by the so called *fake projective planes*, smooth surfaces F with $H^1(F, \mathbb{Z}) = 0$ and $H^2(F, \mathbb{Z}) \cong \mathbb{Z}$ (cf. [BHPVdV04, V.1]). For these we can take $S = F$. All of these are quotients of the complex unit ball (this is a rather difficult result of [Yau77]) hence have a large fundamental group.

Most surfaces with $q = p_g = 0$ are not simply connected (cf. [BHPVdV04, VII.10]). The only known simply connected examples have large Picard number.

With no basis whatsoever, other than the lack of such examples, I suggest that this may be a general phenomenon. The following precise form is the one needed in the classification of Seifert structures on simply connected rational homology spheres to be studied in Section 7.

Conjecture 42. *Let S be a projective surface with quotient singularities such that $H^2(S, \mathbb{Q}) \cong \mathbb{Q}$ and S^0 is simply connected. Then S is rational.*

For surfaces with $H_2(S, \mathbb{Q}) = \mathbb{Q}$, one can make (34) even more explicit.

Corollary 43. *Let S be a normal, projective surface with rational singularities P_i with links M_i . Assume that $H_1(S, \mathbb{Z}) = 0$ and $H_2(S, \mathbb{Q}) = \mathbb{Q}$. Then the following conditions are equivalent*

- (1) $H_1(S^0, \mathbb{Z}) = 0$.
- (2) $\text{Weil}(S) \cong \mathbb{Z}$.
- (3) *Each $H^2(M_i, \mathbb{Z})$ is cyclic, their orders m_i are pairwise relatively prime and there is a Weil divisor E which generates the $H^2(M_i, \mathbb{Z})$ for every i .*
- (4) *There is a Weil divisor D with $(D^2) = 1/\prod m_i$.*
- (5) *There is a Cartier divisor H and a Weil divisor D such that $(D \cdot H) = 1$ and $(H^2) = \prod m_i$.*

Proof. By (40) and (38), $\text{Weil}(S) \cong H_2(S, \mathbb{Z})$. Thus the equivalence of the first 3 are clear from the sequence (32.6). The last two conditions are reformulations of (34.2). \square

The vanishing of the abelian orbifold fundamental group is also given by a simple algebraic condition:

Corollary 44. *Let S be a normal, projective, rational surface with rational singularities P_i with links M_i . Assume that $H_1(S, \mathbb{Z}) = 0$ and $H_2(S, \mathbb{Q}) = \mathbb{Q}$. Then $H_1^{\text{orb}}(S, \sum_j (1 - \frac{1}{m_j}) D_j) = 0$ iff*

- (1) $\text{Weil}(S) \cong \mathbb{Z}$,
- (2) *the m_j are pairwise relatively prime, and*
- (3) *m_j is relatively prime to the degree $\deg D_j \in \mathbb{Z} \cong \text{Weil}(S)$ for every j .* \square

4. THE TOPOLOGY OF 5-DIMENSIONAL SEIFERT BUNDLES

The aim of this section is to describe the (topological) cohomology groups of a Seifert bundle over a rational surface in terms of invariants of the base orbifold.

The following is a straightforward generalization of the computation of the fundamental group of 3-dimensional Seifert bundles, cf. [Sei32].

Proposition 45. [HS91, 5.7], [Kol04b, Prop.50] *Let $f : L \rightarrow (X, \Delta)$ be a Seifert bundle such that L is smooth. There is an exact sequence*

$$\pi_1(\mathbb{C}^*) \rightarrow \pi_1(L) \rightarrow \pi_1^{orb}(X, \Delta) \rightarrow 1. \quad \square$$

The determination of $\pi_1(L)$ seems rather tricky in general, but its abelianization is fully computable. The case when $H_1(X^0, \mathbb{Z}) = 0$ is especially easy to state.

Proposition 46. [OW75, 4.6] *Let X be a complex manifold such that $H_1(X, \mathbb{Z}) = 0$ and let $D_1, \dots, D_n \subset X$ be smooth divisors intersecting transversally. For any divisor B , $H_1(Y(B, \sum \frac{b_i}{m_i} D_i), \mathbb{Z})$ is given by generators k, g_1, \dots, g_n and relations*

- (1) $m_i g_i + b_i k = 0$ for $i = 1, \dots, n$, and
- (2) $k([B] \cap \eta) - \sum g_i([D_i] \cap \eta) = 0$ for every $\eta \in H_2(X, \mathbb{Z})$. \square

Remark 47. The assumptions imposed on X in (46) seem somewhat restrictive, but the result can be used to compute the first homology of any smooth Seifert \mathbb{C}^* -bundle. Indeed, if $f : Y \rightarrow (X, \Delta)$ is any Seifert \mathbb{C}^* -bundle, then let $Z \subset X$ denote the union of all singular points of X and of all singular points of $\cup D_i$. Z has codimension at least 2 in X , so $\pi_1(Y) = \pi_1(Y \setminus f^{-1}(Z))$. Thus $H_1(Y, \mathbb{Z}) = H_1(Y \setminus f^{-1}(Z), \mathbb{Z})$ can be computed by applying (46) to $X \setminus Z$.

Corollary 48. *With notation and assumptions as in (46), set $Y := Y(B, \sum \frac{b_i}{m_i} D_i)$. As noted in (27), $M(\Delta) \cdot c_1(Y/X) \in H^2(X, \mathbb{Z})$ is well defined and thus can be written as $M(\Delta) \cdot c_1(Y/X) = d(Y) \cdot U$ where $d(Y) \in \mathbb{Z}$ and $U \in H^2(X, \mathbb{Z})$ is primitive.*

Then $d(Y) \cdot k = 0$ in $H_1(Y, \mathbb{Z})$.

Proof. Multiply (46.2) by $M(\Delta)$ and use that

$$M(\Delta)g_i = \frac{M(\Delta)}{m_i}(m_i g_i) = -\frac{M(\Delta)}{m_i}b_i k$$

to rewrite it as

$$((M(\Delta) \cdot c_1(Y/X)) \cap \eta) \cdot k = 0 \quad \text{for every } \eta \in H^2(X, \mathbb{Z}).$$

Since $M(\Delta) \cdot c_1(Y/X) = d(Y) \cdot (\text{primitive class})$, there is an η such that $(M(\Delta) \cdot c_1(Y/X)) \cap \eta = d(Y)$. Thus $d(Y) \cdot k = 0$. \square

The following is a very convenient property of orbifolds with $H_1^{orb}(X, \Delta) = 0$.

Proposition 49. [Kol04b, 53] *Yet (X, Δ) be an orbifold and assume that $H_1^{orb}(X, \Delta) = 0$. Then a Seifert \mathbb{C}^* -bundle $f : Y \rightarrow (X, \Delta)$ is uniquely determined by its Chern class $c_1(Y/X) \in H^2(X, \mathbb{Q})$.* \square

The main result of this section is the following:

Theorem 50. *Let $f : L^5 \rightarrow (S, \Delta = \sum_i (1 - \frac{1}{m_i}) D_i)$ be a smooth Seifert bundle over a projective surface with rational singularities. Assume that $H_1(L, \mathbb{Q}) = 0$ and $H_1^{orb}(S, \Delta) = 0$. Set $s = \text{rank } H^2(S, \mathbb{Q})$.*

(1) The cohomology groups $H^i(L, \mathbb{Z})$ are

$$\begin{array}{ccccccc} H^0 & H^1 & H^2 & H^3 & H^4 & H^5 \\ \mathbb{Z} & 0 & \mathbb{Z}^{s-1} + \mathbb{Z}/d & \mathbb{Z}^{s-1} + \sum_i (\mathbb{Z}/m_i)^{2g(D_i)} & \mathbb{Z}/d & \mathbb{Z}. \end{array}$$

(2) Here d is the largest natural number such that

- (a) $M(\Delta) \cdot c_1(Y/S) \in \text{Weil}(S)$ is divisible by $d = d_w$, or equivalently
- (b) $M(S, \Delta) \cdot c_1(Y/S) \in \text{Pic}(S)$ is divisible by $d = d_p$.

Proof. The cohomology groups $H^i(L, \mathbb{Z})$ are computed by a Leray spectral sequence whose E_2 term is

$$E_2^{i,j} = H^i(S, R^j f_* \mathbb{Z}_L) \Rightarrow H^{i+j}(L, \mathbb{Z}).$$

Every fiber of f is S^1 , so $R^2 f_* \mathbb{Z}_L = 0$ and the only interesting higher direct image is $R^1 f_* \mathbb{Z}_L$. We start by computing it.

Lemma 51. *Let $L^5 \rightarrow (S, \Delta = \sum_i (1 - \frac{1}{m_i}) D_i)$ be a Seifert bundle over a surface. Assume that $H_1^{orb}(S, \Delta) = 0$. Then*

- (1) $H^1(S, R^1 f_* \mathbb{Z}_L)$ is torsion.
- (2) $H^2(S, R^1 f_* \mathbb{Z}_L) \cong \mathbb{Z}^s + \sum_i (\mathbb{Z}/m_i)^{2g(D_i)}$ where $s = \text{rank } H^2(S, \mathbb{Q})$,
- (3) $H^3(S, R^1 f_* \mathbb{Z}_L) = 0$.
- (4) $H^4(S, R^1 f_* \mathbb{Z}_L) = \mathbb{Z}$.

Proof. By [Kol04b, 48] there is an exact sequence

$$0 \rightarrow R^1 f_* \mathbb{Z}_L \xrightarrow{\tau} \mathbb{Z}_S \rightarrow Q \rightarrow 0 \quad (51.5)$$

where Q in turn sits in another exact sequence

$$0 \rightarrow \sum_i \mathbb{Z}_{P_j} / n_j \rightarrow Q \rightarrow \sum_i \mathbb{Z}_{D_i} / m_i \rightarrow 0,$$

where $P_j \in S$ are the singular points. Thus $H^0(S, Q)$ is torsion and $H^i(S, Q) = \sum_i H^i(D_i, \mathbb{Z}/m_i)$ for $i \geq 1$. Putting these into the long cohomology sequence of (51.5) we get (1) and (4) right away.

$H^1(S, \mathbb{Z}) = H^1(S^0, \mathbb{Z}) = 0$ since $H_1^{orb}(S, \Delta) = 0$, and by (32.1) this implies that $H^3(S, \mathbb{Z}) = 0$. The remaining sequence is

$$\begin{aligned} 0 \rightarrow \sum_i H^1(D_i, \mathbb{Z}/m_i) &\rightarrow H^2(S, R^1 f_* \mathbb{Z}_L) \rightarrow H^2(S, \mathbb{Z}) \\ &\rightarrow \sum_i H^2(D_i, \mathbb{Z}/m_i) \rightarrow H^3(S, R^1 f_* \mathbb{Z}_L) \rightarrow 0. \end{aligned}$$

$H^2(S, \mathbb{Z}) \rightarrow \sum_i H^2(D_i, \mathbb{Z}/m_i)$ is surjective by (36), which gives the rest. \square

In the Leray spectral sequence $H^i(S, R^j f_* \mathbb{Z}_L) \Rightarrow H^{i+j}(L, \mathbb{Z})$ the E_2 term is

$$\begin{array}{cccccc} \mathbb{Z} & (\text{torsion}) & \mathbb{Z}^s + \sum_i (\mathbb{Z}/m_i)^{2g(D_i)} & 0 & \mathbb{Z} \\ \mathbb{Z} & 0 & \mathbb{Z}^s & 0 & \mathbb{Z}. \end{array}$$

The spectral sequence degenerates at E^3 and we have only two nontrivial differentials

$$\delta_0 : E_2^{0,1} \rightarrow E_2^{2,0} \quad \text{and} \quad \delta_2 : E_2^{2,1} \rightarrow E_2^{4,0}.$$

By [Kol04b, 44], the image of $\delta_0 : \mathbb{Z} \rightarrow \mathbb{Z}^s$ is generated by $M(S, \Delta) \cdot c_1(Y/S)$. It is nonzero since $H_1(L, \mathbb{Q}) = 0$ by assumption. Thus δ_2 is also nonzero, either by multiplicativity or by noting that otherwise Poincaré duality would fail on L .

Hence the E_3 term is

$$\begin{array}{ccccc} 0 & (\text{torsion}) & \mathbb{Z}^{s-1} + \sum_i (\mathbb{Z}/m_i)^{2g(D_i)} & 0 & \mathbb{Z} \\ \mathbb{Z} & 0 & \mathbb{Z}^{s-1} + \mathbb{Z}/d_p & 0 & (\text{torsion}), \end{array}$$

where d_p is as defined in (50.2.b). The torsion in $E_3^{i,0}$ injects into the torsion in $H^i(L, \mathbb{Z})$.

In the notation of (48), $H_1(L, \mathbb{Z})$ is generated by k and its order divides d_w as defined in (50.2.a). (We need to apply (48) to S^0 instead of S , and $H^2(S^0, \mathbb{Z}) \cong H_2(S, \mathbb{Z})$. Since $H_2(S, \mathbb{Z})/\text{Weil}(S)$ is torsion free by (38), divisibility in $H_2(S, \mathbb{Z})$ is the same as divisibility in $\text{Weil}(S)$.)

Thus d_p divides d_w and we need to prove that $d_p = d_w$.

$\text{Weil}(S)/\text{Pic}(S)$ is $M(S)$ -torsion and $M(S, \Delta) \cdot c_1(Y/S) = K \cdot M(\Delta) \cdot c_1(Y/S)$ for some $K|M(S)$ by (27). Thus divisibility by a number relatively prime to $M(S)$ is unchanged when we go from $\text{Weil}(S)$ to $\text{Pic}(S)$. Therefore we only need to prove that $M(\Delta) \cdot c_1(Y/S) \in \text{Weil}(S)$ is not divisible by any prime divisor of $M(S)$.

Assuming the converse, let p be such a prime and write $M(S) = p^a p'$ where p does not divide p' . Assume that $M(\Delta) \cdot c_1(Y/S) = p \cdot A$ for some $A \in \text{Weil}(S)$. If $p \nmid M(\Delta)$ then $M(\Delta) \cdot c_1(Y/S)$ has order divisible by p^a in some $\text{Weil}(x, X)$ by (30) and so it can not be divisible by p . If $p|M(\Delta)$ then $M(\Delta) \cdot c_1(Y/S) = p \cdot A$ can be rearranged to be

$$\sum_{i:p|m_i} a_i D_i = p \cdot (\text{integral Weil divisor}),$$

and not all the a_i are divisible by p . By (37) this would imply $H_1^{orb}(S, \Delta) \neq 0$, a contradiction. \square

5. LOG DEL PEZZO SURFACES WITH NONRATIONAL BOUNDARY

We see from (50) that torsion in $H_2(L, \mathbb{Z})$, or equivalently, torsion in $H^3(L, \mathbb{Z})$ is connected with higher genus curves in the branch divisor Δ of $(S, \Delta = \sum (1 - \frac{1}{m_i}) D_i)$.

The aim of this section is to study the cases when Δ has an irreducible component of genus at least 1. This turns out to be a strong restriction, especially if the m_i are not very small. A result of this type is a special case of a general principle.

52 (Ascending chain conditions). Assume that $(S, \Delta = \sum_{i=0}^n a_i D_i)$ is a pair where we only assume that $-(K_S + \Delta)$ is nef but we consider the case when $a_0 = 1$ and D_0 has geometric genus ≥ 1 . The adjunction formula (with a little extra work for the singularities) says that

$$\deg K_{D_0} \leq D_0 \cdot (K_S + D_0) \leq D_0 \cdot \left(- \sum_{i=1}^n a_i D_i \right) \leq 0.$$

Thus D_0 is elliptic and we also get that $\Delta = D_0$.

The ascending chain condition principle predicts that all this should also work if a_0 is close to 1, and [Kol94, sec.5] implies this for $a_0 \geq 41/42$. This is quite surprising at first sight since we work with singular surfaces, and the various intersection numbers are only rational numbers.

Here we are in a rather special situation, and we get better bounds with careful case analysis.

Definition 53. A *log Del Pezzo surface* is a pair (S, Δ) where

- (1) S is a normal, projective surface,
- (2) $\Delta := \sum a_i D_i$ is a linear combination of distinct irreducible divisors with $0 \leq a_i \leq 1$, and
- (3) $-(K_S + \Delta)$ is ample. (That is, a suitable multiple of it is an ample Cartier divisor.)

Thus a 2-dimensional log Fano orbifold is a log Del Pezzo surface.

The \mathbb{Q} -divisor Δ is called the *boundary*. In the orbifold case the boundary is of the form $\Delta = \sum (1 - \frac{1}{m_i}) D_i$, where the m_i are natural numbers. Such a boundary is sometimes called *standard*.

Ideally one would like to have a classification of all log Del Pezzo surfaces, but this seems quite out of reach. Some kind of rough structure theorems are given in [KM99, Sho00].

Proposition 54. *Let $(S, \Delta = \sum_{i=0}^n a_i D_i)$ be a log Del Pezzo surface. Assume that D_0 has geometric genus ≥ 1 and $a_0 \geq 1/2$. Then D_i is rational for $i \geq 1$ and*

- (1) $g(D_0) = 1$ if $a_0 \geq 5/6$,
- (2) $g(D_0) \leq 2$ if $a_0 \geq 3/4$,
- (3) $g(D_0) \leq 4$ if $a_0 \geq 2/3$.

The proof of this is an easy application of the minimal model program for rational surfaces. In general, it is difficult to run the minimal model program backwards, and it seems complicated to get a full classification, except when a_0 is close to one.

Proposition 55. *Notation and assumptions as in (54). Assume in addition that $a_0 \geq 11/12$ and $a_i \geq 1/2$ for every i . Then S is a Del Pezzo surface with Du Val singularities and $\Delta = a_0 D_0$. The Picard number of S is at most 8.*

Del Pezzo surfaces of Picard number 1 and with Du Val singularities are classified in [Fur86]. Adding \mathbb{P}^2 and the quadric cone, one gets 29 types. (Some types correspond to more than 1 surface, I do not know how many there are up to isomorphism.) By the results of [MZ88], or by checking the conditions of (43), we obtain the following.

Proposition 56. *Let $(S, (1 - \frac{1}{m})D)$ be a log Del Pezzo surface of Picard number 1, with $m \geq 12$ and $g(D) \geq 1$. If $H_1(S^0, \mathbb{Z}) = 0$ then S is one of the following:*

- (1) \mathbb{P}^2 . Here $\text{Pic}(S) = \text{Weil}(S)$ and it is generated by the class of a line.
- (2) Q , the quadric cone. Here $\text{Pic}(S) = 2 \cdot \text{Weil}(S)$, the latter generated by the lines through the vertex of the cone.
- (3) $\mathbb{P}^2(1, 2, 3)$, the weighted projective plane with weights 1, 2, 3. Here $\text{Pic}(S) = 6 \cdot \text{Weil}(S)$, the latter generated by the line $(x = 0)$ where x is the weight 1 coordinate.
- (4) S_5 , a degree 5 Del Pezzo surface with a single point of index 5. This is obtained by blowing up a flex of a smooth cubic 4 times and then contracting the tangent line and the first 3 exceptional curves. If G is the equation of the cubic and L the equation of the flex tangent then G/L^3 lifts to a rational function on S_5 which has a 5 fold pole along the unique line of the surface which passes through the singular point. The line generates $\text{Weil}(S_5)$ and $\text{Pic}(S_5) = 5 \cdot \text{Weil}(S_5)$.

In all of these cases, D is a smooth member of $-K_S$ and it has genus 1. \square

The following two examples show that for $m \leq 11$ there are other cases as well.

Example 57. On $\mathbb{P}(1, 2, 3)$ consider the divisor $\Delta = \frac{1}{2}(x = 0) + \frac{10}{11}(x^6 + y^3 + z^2 = 0)$. The surface S is as in (56) but Δ is different. Also, the coefficient $\frac{10}{11}$ can be replaced with anything smaller.

Example 58. Let S be a degree 1 Del Pezzo surface where $|-K_S|$ has a member D_1 of type II^* (also denoted by \tilde{E}_8) on Kodaira's list (cf. [BHPVdV04, V.7]). This has a unique point P of multiplicity 11. Blow it up $S' \rightarrow S$ and contract the birational transform D'_1 of D_1 to get $S' \rightarrow S^*$, a log Del Pezzo surface with Picard number 1.

Let D_0 be a smooth elliptic member of $|-K_S|$ and D_0^* its birational transform on S^* . Then

$$\frac{10}{11}D_0 + \frac{1}{11}D_1 \equiv -K_S \quad \text{and} \quad \frac{10}{11}D'_0 + \left(1 - \frac{1}{11}\right)D'_1 \equiv -K_{S'}.$$

Thus

$$-K_{S^*} \equiv \frac{10}{11}D_0^* \quad \text{and} \quad -(K_{S^*} + \left(1 - \frac{1}{10}\right)D_0^*) \quad \text{is ample.}$$

This gives an example $(S^*, (1 - \frac{1}{10})D_0^*)$ which is not part of the main series.

59 (Proof of (54)). Let $g : S' \rightarrow S$ be the minimal resolution of S . $K_{S'} \equiv g^*K_S - (\text{effective divisor})$, thus $-K_{S'} \equiv a_0D'_0 + (\text{big effective divisor})$ where $D'_0 \subset S'$ is the birational transform of D_0 . Since S' is rational, there is a morphism $h : S' \rightarrow S^m$ where S^m is either \mathbb{P}^2 or a minimal ruled surface. Only rational curves are contracted by h , so $-K_{S^m} \equiv a_0D_0^m + (\text{big effective divisor})$ where $D_0^m = h(D'_0)$.

If $S^m \cong \mathbb{P}^2$ and $\deg D_0^m \geq 4$, then $4a_0 \leq a_0 \deg D_0^m < 3$ implies that $a_0 < 3/4$. Thus if $a_0 \geq 3/4$ then $D_0^m \subset \mathbb{P}^2$ is a smooth degree 3 curve.

Similarly, if $S^m = \mathbb{P}^1 \times \mathbb{P}^1$, the smooth quadric, we get that D_0^m is a smooth elliptic curve cut out by another quadric if $a_0 \geq 2/3$.

The remaining possibility is that $S^m \cong \mathbb{F}_n$, the minimal ruled surface with a section E of selfintersection $-n$ and $n \geq 2$. Let F denote a fiber of the projection to \mathbb{P}^1 . Then $-K_{\mathbb{F}_n} = 2E + (n+2)F$. If $|aE + bF|$ has a nonrational member then $a \geq 2$ and $b \geq na$ and so in our case

$$2E + (n+2)F - a_0(2E + 2nF) = (2 - 2a_0)E + (n+2 - 2na_0)F$$

is effective and big. Thus $n+2 > 2na_0$ and for $a_0 \geq 2/3$ this gives $n \leq 5$. We also get that $a_0 \leq 5/6$ for $n \geq 3$.

If $a \geq 3$ then the coefficient of E is $\leq 2 - 3a_0$, thus again $a_0 < 2/3$. Hence we need to enumerate all cases when $n = 2, 3, 4, 5$ and $C \in |2E + (2n+c)F|$ for some $c \geq 0$. As before, we get that $(2 - 2a_0)E + (n+2 - (2n+c)a_0)F$ is effective and big which implies that $a_0 < (n+2)/(2n+c)$.

If $n = 2$ and $c = 0$ then D_0 is elliptic. In all other cases we get that $a_0 < 5/6$ and $a_0 \geq 4/5$ only if $n = 3, c = 0$. Furthermore, $a_0 \geq 3/4$ in one additional case only, when $n = 2, c = 1$. \square

Example 60. Let $F_n \subset \mathbb{P}^{n+1}$ be the cone over the rational normal curve of degree n in \mathbb{P}^n . It can be also realized as $F_n = \mathbb{P}(1, 1, n)$. $\text{Weil}(F_n)$ is generated by the lines L . $K_{F_n} \sim -(n+2)L$ and $\text{Pic}(F_n)$ is generated by the hyperplane class which is nL .

Let $C \subset F_n$ be a smooth intersection of F_n with a quadric. Thus $C \in |2nL|$ and $g(C) = n - 1$. The surface $(F_n, (1 - \frac{1}{m})C)$ is log Del Pezzo and $g(C) > 1$ in the following cases.

- (1) $n = 3$ and $m \leq 5$ with $g(C) = 2$

- (2) $n = 4$ and $m \leq 3$ with $g(C) = 3$.
- (3) $n = 5$ and $m \leq 3$ with $g(C) = 4$.
- (4) $n \geq 3$ and $m = 2$ with $g(C) = n - 1$

Example 61. On $\mathbb{P}(1, 2, 5)$ consider a general member $C \in |\mathcal{O}(10)|$. C is smooth, has genus 2 and the pair $(\mathbb{P}(1, 2, 5), \frac{3}{4}C)$ is log Del Pezzo.

Proposition 62. *Let $(S, (1 - \frac{1}{m})D_0 + \sum a_i D_i)$ be a log Del Pezzo surface. Assume that $g(D_0) \geq 2$, $m \geq 5$ and $a_i \geq \frac{1}{2}$. Then $S \subset \mathbb{P}^4$ is the cone over the rational cubic and D_0 is the intersection of S with a quadric.*

There is a unique such pair $(S, (1 - \frac{1}{5})D_0)$ for every genus 2 curve D_0 .

Proof. From the case analysis in (59) we see that the minimal resolution of S dominates \mathbb{F}_3 and does not dominate any other minimal rational surface. Any one point blow up of \mathbb{F}_3 dominates either \mathbb{F}_2 or \mathbb{F}_4 , thus the minimal resolution of S is \mathbb{F}_3 .

The first possibility is that S is the cone over the rational cubic and then D is the intersection of S with a quadric by the cases analysis.

The second possibility is that $S = \mathbb{F}_3$. Using the notation of (59) we get that $(1 - \frac{1}{m})D \leq -K_S = 2E + 5F$. Since $m \geq 5$, this implies that $D \leq 2E + 6F$. The condition $g(D) \geq 2$ implies that in fact $D = 2E + 6F$. Then $-(K_S + (1 - \frac{1}{m})D) \leq \frac{2}{5}E + \frac{1}{5}F$ and there is no room for any other boundary curves. Finally $-(K_S + (1 - \frac{1}{5})D)$ has negative intersection number with E , a contradiction.

Let D be any genus 2 curve and $f : D \rightarrow \mathbb{P}^1$ the canonical map. Then $f_*\mathcal{O}_D = \mathcal{O}_{\mathbb{P}^1} + \mathcal{O}_{\mathbb{P}^1}(-3)$. This provides the unique embedding of D into \mathbb{F}_3 , the projectivization of $\mathcal{O}_{\mathbb{P}^1} + \mathcal{O}_{\mathbb{P}^1}(-3)$. \square

63 (Proof of (55)). As we noted during the proof of (54), $a_0 \geq 5/6$ implies that $S^m = \mathbb{P}^2, \mathbb{P}^1 \times \mathbb{P}^1$ or \mathbb{F}_2 .

S' is obtained from S^m by repeatedly blowing up points $S' = S_n \rightarrow S_{n-1} \rightarrow \dots \rightarrow S^m$.

Let us look at the first blow up $h : S_j \rightarrow S_{j-1}$ where we blow up a point p not on the birational transform of D_0 . Let $E \subset S_j$ denote the exceptional curve of the blow up. On S_j we can write

$$-K_{S_j} \equiv a_0 D_0^j + \alpha E + \Delta_j^*,$$

where $\alpha \geq 0$, Δ_j^* is effective and its support does not contain E . By our assumption, E is disjoint from D_0^j . This implies that

$$(E \cdot \Delta_j^*) \geq -(K_S \cdot E) - \alpha(E \cdot E) = 1 + \alpha \geq 1,$$

which in turn gives that

$$-K_{S_{j-1}} \equiv a_0 D_0^{j-1} + h_*(\Delta_j^*) \quad \text{and} \quad \text{mult}_p h_*(\Delta_j^*) \geq 1.$$

$1/(1 - a_0)h_*(\Delta_j^*)$ is numerically equivalent to $-K_{S_{j-1}}$ and it has multiplicity at least $1/(1 - a_0) \geq 12$ at p . This is impossible by (64).

Thus as we go from S^m to S' , we can blow up at most 8 points on \mathbb{P}^2 (or at most 7 points on $\mathbb{P}^1 \times \mathbb{P}^1$ or \mathbb{F}_2) and all the blow ups occur on the birational transform of D_0 . Hence S is a Del Pezzo surface with Du Val singularities and its Picard number is at most 9.

Finally, if we had any other curve D_i in Δ , then D_i would occur with coefficient at least $1/2$ in $(1 - \epsilon - a_0)K_S$, so we would obtain a curve with coefficient bigger than

6 in an effective divisor numerically equivalent to $-K_S$. This is again impossible by (64). \square

Lemma 64. *Let Y be one of the surfaces \mathbb{P}^2 , $\mathbb{P}^1 \times \mathbb{P}^1$ or \mathbb{F}_2 , $C \subset Y$ a smooth cubic. Let $f : Z \rightarrow Y$ be obtained by repeatedly blowing up points of C . Let $D \subset Z$ be a divisor numerically equivalent to $-K_Z$. Then*

- (1) $\text{mult}_z D \leq 11$ for every point $z \in Z$.
- (2) $\text{mult}_A D \leq 6$ for every curve $A \subset Z$.

Proof. Let $Z = Y_n \rightarrow Y_{n-1} \rightarrow \cdots \rightarrow Y_j = Y$ be the sequence of blow ups where we set $Y = Y_0$ if $Y = \mathbb{P}^2$ and $Y = Y_1$ if $Y = \mathbb{P}^1 \times \mathbb{P}^1, \mathbb{F}_2$.

The push forward of D to any of the Y_i is again numerically equivalent to $-K_{Y_i}$ and the multiplicity does not change unless z or A is contained in the exceptional set. Thus by induction on n it is sufficient to consider the case when $g : Z \rightarrow Z' := Y_{n-1}$ is the blow up of a point p and z or A is contained in the exceptional curve E of g . Set $D' := g_*(D)$, then $D = g^*(D') + (\text{mult}_p D' - 1)E$.

If $n \geq 10$ then $(-K_Z)^2 = 9 - n < 0$ and so the only divisor numerically equivalent to $-K_Z$ is C itself which is smooth.

Let $C' \subset Z'$ denote the birational transform of C . Then $\text{mult}_p D' \leq (C' \cdot D') = 9 - (n - 1)$, hence if $n \geq 4$ then $\text{mult}_p D' \leq 6$ and so $\text{mult}_E D \leq 5$ and $\text{mult}_z D \leq 11$ for every $z \in E$.

We are left with the cases when $n \leq 3$, that is, we blow up at most 3 points. In these cases Z is a toric variety. By degenerating D using the torus action and noting that multiplicity is upper semi continuous in degenerations, we are reduced to the case when D is fixed by the torus. These cases are easy to enumerate by hand. \square

6. LOG DEL PEZZO SURFACES WITH $H_1^{\text{orb}}(S) = 0$

The aim of this section is to classify Del Pezzo surfaces with cyclic Du Val singularities satisfying $H_1^{\text{orb}}(S) = 0$.

There are two ways to proceed.

65 (Traditional approach). The minimal resolution of a Del Pezzo surface S with Du Val singularities is obtained from \mathbb{P}^2 and a smooth cubic curve $C \subset \mathbb{P}^2$ by blowing up $m \leq 8$ points on the cubic.

In the smooth case, the m points are different and we get only one family for each m . In the singular case, the deformation type of the resulting surface is determined by the following data:

- (1) Any number of the m points may coincide, and we have to mark all coinciding point pairs.
- (2) We have to mark triplets of points that are on a line. (Keep in mind that if 3 points coincide, they are also on a line if this point is an inflection point of the cubic.)
- (3) We have to mark sextuplets of points that are on a conic. (Again, this can happen with 6 points coinciding at a 6-torsion point.)

It is clear that the number of cases is finite, but it would be quite tedious to get a complete list.

The main problem with this approach is that I do not see any efficient way to compute $H_1^{\text{orb}}(S)$. This seems to be nonzero for the majority of all cases.

66 (Minimal model approach). Here we start with a Del Pezzo surface S with Du Val singularities and run the minimal model program to get $g : S \rightarrow S^m$. We classify S using the following observations.

- (1) $H_1^{orb}(S)$ and $\pi_1^{orb}(S)$ are unchanged during the minimal model program; see (67).
- (2) S^m is either $\mathbb{P}^1 \times \mathbb{P}^1$ or one of the 4 surfaces listed in (56); see (69).
- (3) For a fixed S^m , the deformation types of all possible surfaces $S \rightarrow S^m$ are classified by sequences of natural numbers $m_1 \leq \dots \leq m_k$ such that $\sum m_i < (K_{S^m})^2$; see (68).
- (4) Many of the surfaces have representations over different surfaces S^m , but these are not too hard to understand; see (70).

67 (Contractions with Du Val singularities). Let T be any projective surface with Du Val singularities and $f : T \rightarrow U$ a birational contraction with connected exceptional curve $E \subset T$ such that $-K_T$ is f -ample. These are easy to classify, see [KM99, 3.3]. One gets that $u := f(E) \in U$ is a smooth point and f can be described as follows.

Let $u \in C \subset U$ be a smooth curve germ. Fix a number m and blow up $u \in C$ repeatedly m -times. We get m exceptional curves. One of these is a -1 -curve, the other $m - 1$ form a chain of -2 -curves. This chain can be contracted to a point of type A_m . (It can be described by local equations as $xy - z^{m+1} = 0$ or as a quotient of \mathbb{C}^2 by the $\mathbb{Z}/(m+1)$ -action $(x, y) \mapsto (\epsilon x, \epsilon^{-1}y)$.) The resulting surface is T . Thus we see that $E \subset T$ is orbisooth, T has a unique singular point $t \in E$ and a neighborhood of E minus t is homotopic to a lense space $S^3/(\mathbb{Z}/m)$ with a disc attached (corresponding to $E \setminus \{t\}$), killing the fundamental group. From this, or using [KM99, 3.3] we conclude that $H_1^{orb}(T) = H_1^{orb}(U)$ and $\pi_1^{orb}(T) = \pi_1^{orb}(U)$.

The deformation type of T is determined by the deformation type of U and the number m .

Definition 68. Let S be a Del Pezzo surface with Du Val singularities and $m_1 \geq \dots \geq m_k \geq 1$ integers. We denote by $B_{m_1, \dots, m_k} S$ any surface obtained as follows.

Pick any smooth elliptic curve $C \in |-K_S|$ and p_1, \dots, p_k distinct points on C . Then perform a blow up type m_i at p_i .

All such surfaces form one deformation type. Furthermore, S and the deformation type determine the numbers m_i . Indeed, the numbers $m_i \geq 2$ can be read off from the singularities and the Picard number determines k .

The canonical class of $B_{m_1, \dots, m_k} S$ is nef and big iff $\sum m_i < (K_S^2)$. If this holds then $B_{m_1, \dots, m_k} S$ is a Del Pezzo surface for general choice of the points p_i .

69 (The minimal models S^m). If we start with a Del Pezzo surface with (cyclic) Du Val singularities S , then all steps of the minimal model program yield Del Pezzo surfaces with (cyclic) Du Val singularities. The program eventually stops, and we end up with $S \rightarrow S^m$ where S^m does not have birational contractions. This can happen in two ways

- (1) S^m has Picard number 1. If, in addition, $H_1^{orb}(S) = 0$ and S has cyclic Du Val singularities then $H_1^{orb}(S^m) = 0$ and by (56) S^m is one of the 4 surfaces

$$\mathbb{P}(1, 2, 3), Q, \mathbb{P}^2 \quad \text{or} \quad S_5.$$

- (2) S^m has Picard number 2 and it has two different conic bundle structures. For S^m smooth, this happens only for $\mathbb{P}^1 \times \mathbb{P}^1$. A list of the singular cases

is given in [MZ93]. They are either not simply connected or have noncyclic singularities.

We can now state the main classification theorem of the section:

Theorem 70. *There are 93 deformation types of Del Pezzo surfaces with cyclic Du Val singularities satisfying $H_1^{orb}(S) = 0$. These are*

- (1) $B_{m_1, \dots, m_k} \mathbb{P}(1, 2, 3)$ for $\sum m_i < 6$ and $m_i \geq 2$,
- (2) $B_{m_1, \dots, m_k} Q$ for $\sum m_i < 8$ and $m_i \geq 2$,
- (3) $B_{m_1, \dots, m_k} \mathbb{P}^2$ for $\sum m_i < 9$ and $m_i \geq 2$,
- (4) $B_{m_1, \dots, m_k} \mathbb{P}^1 \times \mathbb{P}^1$ for $\sum m_i < 8$,
- (5) $S_5, B_3 S_5, B_4 S_5$ and $B_1 \mathbb{P}^2$.

All these satisfy $\pi_1^{orb}(S) = 0$.

Proof. We know that all of the surfaces are of the form $B_{m_1, \dots, m_k} S$ where S is one of the 5 surfaces listed in (69). First we prove that if $m_k = 1$ and $S \neq \mathbb{P}^1 \times \mathbb{P}^1$ then the resulting surface is somewhere else on the list. This follows from the easy isomorphisms:

$$B_1 \mathbb{P}(1, 2, 3) \cong B_3 Q, B_1 Q \cong B_2 \mathbb{P}^2, B_{m,1} \mathbb{P}^2 \cong B_m \mathbb{P}^1 \times \mathbb{P}^1, B_1 S_5 \cong B_4 \mathbb{P}^2.$$

This implies that every Del Pezzo surface with cyclic Du Val singularities satisfying $H_1^{orb}(S) = 0$ is either on the list or it is $B_{m_1, \dots, m_k} S_5$ for $\sum m_i < 5$ and $m_i \geq 2$. There are only 5 such cases. For two of these, $m_k = 2$. These are eliminated by the harder isomorphism

$$B_2 S_5 \cong B_4 Q.$$

The easiest way to see this is to notice that these blow ups give a cubic surface, whose explicit model is described in [Hen11, Species XIV, p.79]. In his notation, contracting the line $x = z = 0$ gives Q and contracting the line $y = w = 0$ gives S_5 .

It remains to see that all these are different. Almost all cases are decided by looking at the number of singular points minus the Picard number. For the surfaces in (70.1) we get 1, for those in (70.2) we get 0, (70.3) gives -1 and (70.4) gives ≤ -2 .

It remains to observe that the 4 surfaces in (70.5) are nowhere else on the list. \square

Remark 71. The results of [MZ93], coupled with (70) give a complete list of all deformation types of Del Pezzo surfaces with arbitrary Du Val singularities satisfying $H_1^{orb}(S) = 0$.

By [KM99, 3.10] and [MZ93], there are 5 more such surfaces with Picard number 1, with singularities D_5, E_6, E_7, E_8, E_8 and 3 more with Picard number 2, with 2 conic bundle structures. These have singularities D_4, D_6, D_7 . The blow ups considered in (68) introduce only cyclic quotient singularities, hence there are no isomorphisms between the blow ups, except possibly for the two surfaces of type E_8 . These, however, have $K^2 = 1$, so we can not blow up at all.

7. EINSTEIN METRICS ON SEIFERT BUNDLES

The main impediment to apply (23) is the current shortage of existence results for positive Ricci curvature Kähler–Einstein metrics on orbifolds.

We use the following sufficient algebro–geometric condition. There is every reason to expect that it is very far from being optimal, but it does provide a large selection of good examples.

In this paper we use (72) only for surfaces. The concept *klt* is defined in (73).

Theorem 72. [Nad90, DK01] *Let (X, Δ) be an n -dimensional compact orbifold such that $-(K_X + \Delta)$ is ample. Assume that there is an $\epsilon > 0$ such that*

$$(X, \Delta + \frac{n+\epsilon}{n+1}D) \text{ is klt}$$

for every effective \mathbb{Q} -divisor $D \equiv -(K_X + \Delta)$. Then (X, Δ) has an orbifold Kähler–Einstein metric. \square

Definition 73. Let X be a normal complex space and D a \mathbb{Q} -divisor on X . Assume that mK_X, mD are both Cartier for some $m > 0$ (this is automatic if X is an orbifold). Let $g : Y \rightarrow X$ be any proper birational morphism, Y smooth. Then there is a unique \mathbb{Q} -divisor $D_Y = \sum e_i E_i$ on Y such that

$$K_Y + D_Y \equiv g^*(K_X + D) \quad \text{and} \quad g_* D_Y = D.$$

We say that (X, D) is *klt* (resp. *log canonical*) if $e_i < 1$ (resp. $e_i \leq 1$) for every g and for every i . We say that (X, D) is *canonical* if $e_i \leq 0$ for every g and for every i such that E_i is g -exceptional

It is quite hard to check using the above definition if a pair (X, D) is klt or not. For surfaces, there are reasonably sharp multiplicity conditions which ensure that a given pair (X, D) is klt. These conditions are not necessary, but they seem to apply in most cases of interest to us.

74 (How to check if (X, D) is klt or not?).

Let X be a surface with quotient singularities. Let the singular points be $P_i \in X$ and we write these locally as

$$p_i : B^2 \rightarrow B^2/G_i$$

where B^2 is the unit 2-ball and $G_i \subset GL(2, \mathbb{C})$ a finite subgroup. We may assume that the origin is an isolated fixed point of every nonidentity element of G_i (cf. [Bri66]). Let D be an effective \mathbb{Q} -divisor on X . Then (X, D) is klt if the following three conditions are satisfied.

- (1) (Klt along curves) D does not contain an irreducible component with coefficient ≥ 1 .
- (2) (Klt at smooth points) For every smooth point $P \in X$,
 - (a) either $\text{mult}_P D \leq 1$,
 - (b) or $D = cC + D'$ where C is a curve through P , smooth at P , D' is effective not containing C , and the local intersection number $(C \cdot_P D') < 1$.
 (This follows from [KM98, 4.5 and 5.50].)
- (3) (Klt at singular points) The condition (2) is satisfied for $(B^2, p_i^* D)$ at $P =$ (origin). (This follows from [KM98, 5.20] and the previous case.)

A good illustration of how to use these methods is given by the following example. Some of its conditions seem artificial, but they are satisfied in many cases.

Lemma 75. *Let $S \subset \mathbb{P}^d$ be a Del Pezzo surface (with quotient singularities) of degree d with hyperplane section H . Let $D \subset S$ be a smooth divisor, not passing through any singular point. Assume that*

- (1) $-K_S \equiv aH$ for some $a \in \mathbb{Q}$,
- (2) $D \equiv bH$ for some $b \in \mathbb{Q}$,
- (3) *every singular point is a quotient by a group of order at most d ,*

(4) there is a line $L \subset S$ passing through all the singular points such that $L \equiv \frac{1}{d}H$,

(5) $d(a - (1 - \frac{1}{m})b) < \frac{3}{2+\epsilon}$ and $bd(a - (1 - \frac{1}{m})b) < \frac{3}{2+\epsilon}$.

Then $(S, (1 - \frac{1}{m})C)$ has an orbifold Kähler–Einstein metric.

Proof. Let $D \subset S$ be any effective divisor numerically equivalent to

$$-(K_S + (1 - \frac{1}{m})C) \equiv (a - (1 - \frac{1}{m})b)H.$$

We need to check the conditions of (74) for $(S, (1 - \frac{1}{m})C + \frac{2+\epsilon}{3}D)$.

Since $D \cdot H = d(a - (1 - \frac{1}{m})b) < \frac{3}{2+\epsilon}$, we see that $\frac{2+\epsilon}{3}D$ can not contain any effective curve with coefficient ≥ 1 . For the same reason, D has multiplicity ≤ 1 at every point.

At a point on C , we need that $\frac{2+\epsilon}{3}D \cdot C < 1$. This follows from $bd(a - (1 - \frac{1}{m})b) < \frac{3}{2+\epsilon}$.

Finally consider a singular point, where $p : B^2 \rightarrow S$ has degree $d' \leq d$. Here

$$\text{mult } p^*D \leq (p^*D \cdot p^*L) = d'(D \cdot L) \leq d' \frac{1}{d}(D \cdot H) < \frac{3}{2+\epsilon},$$

if $L \not\subset \text{Supp } D$. If $D = cL + D'$ where $L \not\subset \text{Supp } D'$, then we can use the same estimate for D' to conclude the proof. \square

Example 76 (Existence of Kähler–Einstein metrics). Here we see what (75) gives for the surfaces that we need for (6) and (8.2).

Let us start with the ones in (56).

(1) $(\mathbb{P}^2, (1 - \frac{1}{m})D)$ where D is a smooth cubic. Here $d = 1, a = b = 3$ and we get a Kähler–Einstein metric for $m > 6$.

(2) $(Q, (1 - \frac{1}{m})D)$ where $Q \subset \mathbb{P}^3$ is a quadric cone and D is its intersection with a quadric. Here $d = 2, a = b = 2$ and we get a Kähler–Einstein metric for $m > 5$.

(3) $(\mathbb{P}(1, 2, 3), (1 - \frac{1}{m})D)$ with an embedding $\mathbb{P}(1, 2, 3) \subset \mathbb{P}^6$ as a degree 6 surface and D is a smooth hyperplane section. The curve $((\text{weight } 1 \text{ coordinate}) = 0)$ is the required line. Here $d = 6, a = b = 1$ and we get a Kähler–Einstein metric for $m > 4$.

(4) $(S_5, (1 - \frac{1}{m})D)$ with an embedding $S_5 \subset \mathbb{P}^5$ as a degree 5 surface and D is a smooth hyperplane section. The needed line is constructed in (56). Here $d = 5, a = b = 1$ and we get a Kähler–Einstein metric for $m > 3$.

Finally let us consider some of the surfaces from (60) and (62).

(5) Let $F_n \subset \mathbb{P}^{n+1}$ be the cone over the degree n rational normal curve and $C \subset F_n$ a smooth intersection of F_n with a quadric. Here $d = n, a = 1 + \frac{2}{n}$ and $b = 2$. The conditions of (75) are satisfied and we get a Kähler–Einstein metric in the following cases: $(F_3, \frac{4}{5}C), (F_3, \frac{3}{4}C), (F_4, \frac{2}{3}C)$ and $(F_5, \frac{2}{3}C)$.

Lemma 77. *Let S be a Del Pezzo surface with Du Val singularities, $C \in |-K_S|$ a smooth elliptic curve and D an effective divisor on S numerically equivalent to $\frac{1}{m}(-K_S)$. Assume that $m \geq 9$.*

Then $(S, (1 - \frac{1}{m})C + D)$ is log canonical, thus $(S, (1 - \frac{1}{m})C)$ has an orbifold Kähler–Einstein metric.

Proof. Since $K_S + (1 - \frac{1}{m})C + D$ is numerically zero, being log canonical is preserved under pull backs and push forwards. Thus we can pass first to the minimal resolution, and then go down to $S = \mathbb{P}^2$ or $S = \mathbb{P}^1 \times \mathbb{P}^1$. These are now easy to treat with the estimates of (74). \square

8. SEIFERT BUNDLES AND RATIONAL HOMOLOGY SPHERES

With the aim of constructing new Einstein metrics on homology spheres, one would like to describe all Seifert bundle structures on them.

In view of (44), the following is a more detailed version of (5).

Theorem 78. *Let $f : L \rightarrow (S, \sum(1 - \frac{1}{m_i})D_i)$ be a 5-dimensional Seifert bundle, L smooth.*

- (1) *If L is a rational homology sphere with $H_1(L, \mathbb{Z}) = 0$ then*
 - (a) *S has only cyclic quotient singularities and $H_2(S, \mathbb{Z}) \cong \text{Weil}(S) \cong \mathbb{Z}$,*
 - (b) *the D_i are orbismooth curves, intersecting transversally,*
 - (c) *the m_i are relatively prime to each other and each m_i is relatively prime to $\deg D_i \in H_2(S, \mathbb{Z}) \cong \mathbb{Z}$.*
- (2) *Coverseely, given any $(S, \sum(1 - \frac{1}{m_i})D_i)$ satisfying the above 3 conditions (1.a,b,c), there is a unique (up to orientation) Seifert bundle $f : L \rightarrow (S, \sum(1 - \frac{1}{m_i})D_i)$ such that L is a rational homology sphere with $H_1(L, \mathbb{Z}) = 0$.*
- (3) *L is simply connected iff $\pi_1^{\text{orb}}(S, \sum(1 - \frac{1}{m_i})D_i) = 1$.*
- (4) *L is homeomorphic to S^5 iff $\pi_1^{\text{orb}}(S, \sum(1 - \frac{1}{m_i})D_i) = 1$ and the D_i are all rational.*

This gives a rather complete answer in terms of algebraic geometry. There is, however, one missing piece. If (42) is true then we get further restrictions of S :

Conjecture 79. *Let $f : L \rightarrow (S, \sum(1 - \frac{1}{m_i})D_i)$ be a 5-dimensional Seifert bundle, L smooth. If L is a simply connected rational homology sphere then S is a rational surface.*

Remark 80. A rich source of examples, first considered in [OW75], comes from taking $S = \mathbb{P}^2$. If the D_i are lines and conics intersecting transversally, the m_i are relatively prime to each other and odd for conics, then L is always S^5 .

By allowing higher degree curves for the D_i , we get many examples of simply connected rational homology spheres. However, not all rational homology spheres can be realized. Indeed, the torsion subgroup of $H_2(L, \mathbb{Z})$ is computed by (50). If $H_2(L, \mathbb{Z})$ is torsion, then S has Picard number one, hence any two curves on S intersect. By (25.2) this implies that the m_i are relatively prime to each other. Hence we obtain:

Corollary 81. *Let $f : L \rightarrow (S, \sum(1 - \frac{1}{m_i})D_i)$ be a 5-dimensional Seifert bundle, L smooth. If L is a rational homology sphere with $H_1(L, \mathbb{Z}) = 0$ then*

$$H_2(L, \mathbb{Z}) \cong \sum_i (\mathbb{Z}/m_i)^{2g(D_i)}$$

where the m_i are relatively prime to each other. □

Thus, for instance, $H_2(L, \mathbb{Z})$ can not be $(\mathbb{Z}/p)^2 + (\mathbb{Z}/p^2)^2$.

82 (Proof of (78)). Let us start with $f : L \rightarrow (S, \sum(1 - \frac{1}{m_i})D_i)$ such that L is a rational homology sphere. By (20) and (25), S has only cyclic quotient singularities, the D_i are orbismooth and they intersect transversally. The rest of the conditions follows from (44).

Conversely, start with a surface S with $H_2(S, \mathbb{Z}) \cong \text{Weil}(S) \cong \mathbb{Z}$, with singular points P_i with links M_i with $H^2(M_i, \mathbb{Z}) \cong \mathbb{Z}/n_i$ and orbismooth rational curves D_i of degree d_i and natural numbers m_i .

The n_i are pairwise relatively prime by (43); set $N = \prod n_i$. The m_j are pairwise relatively prime by assumption; set $M = \prod m_j$.

$H_1^{orb}(S, \sum(1 - \frac{1}{m_i})D_i) = 1$ by (44), hence by (49), a Seifert bundle over (S, Δ) is uniquely determined by its Chern class. Furthermore, by (50.2), $H_1(L, \mathbb{Z}) = 1$ iff $c_1(L/S) = \pm \frac{1}{M}[\ell]$ where $[\ell] \in \text{Weil}(S)$ is the positive generator.

If D_j does not pass through P_i then n_i divides $d_j := \deg D_j$ since $\text{Weil}(S) \rightarrow \text{Weil}(P_i, S)$ is surjective by (43) and its kernel is precisely those curves whose degree is divisible by n_i . Hence in this case m_j is relatively prime to n_i . If D_j does pass through P_i then by (25), the multiplicity of the fiber above P_i is divisible by $n_i m_j$. Thus $m(X, \Delta) = NM$.

By (78.1.c) the integers $M, d_j M/m_j$ are relatively prime, thus

$$b_M + \sum \frac{d_j}{m_j} b_j = \frac{1}{M}$$

is solvable in integers b_M, b_j . We can even assume that $0 \leq b_j < m_j$ and b_j is necessarily relatively prime to m_j . We identify $b_M \in \mathbb{Z}$ with a Weil divisor class $B_M \in \text{Weil}(S) \cong \mathbb{Z}$. These data determine a Seifert bundle $f : L \rightarrow (S, \sum(1 - \frac{1}{m_i})D_i)$. (The solution corresponding to $-\frac{1}{M}$ gives a Seifert bundle which differs from this by reversing the orientation of the circles.)

$M \cdot c_1(L/S)$ is a generator of $\text{Weil}(S)$, and it generates the local class groups $\text{Weil}(s, S)$ by (43.3). Thus L is smooth by (30).

L is a rational homology sphere by (50).

Finally, $\pi_1(L) = 1$ iff $\pi_1^{orb}(S, \sum(1 - \frac{1}{m_i})D_i) = 1$ by (45).

L is homeomorphic to S^5 iff it is a simply connected integral homology sphere. By (50), $H_2(L, \mathbb{Z}) = 0$ iff the D_i are rational. Duality gives $H_3(L, \mathbb{Z}) = 0$. \square

We are now ready to prove the main theorems of this paper.

83 (Proof of (4)). Let $f : L \rightarrow (S, \sum(1 - \frac{1}{m_i})D_i)$ be a Seifert bundle with $H_1(L, \mathbb{Z}) = 0$ such that $(S, \sum(1 - \frac{1}{m_i})D_i)$ is a log Del Pezzo orbifold. If $\text{tors } H_2(L, \mathbb{Z}) \neq 0$, then by (50.1), at least one of the curves, say D_0 , is nonrational. By (54), all the others are rational and the relationships between the genus of D_0 and m_0 given in (54) give the cases (4.1–4).

The existence of Kähler–Einstein metrics on the following surfaces was established in (76.5):

$(F_3, \frac{4}{5}C)$, giving $H_2(L, \mathbb{Z}) = (\mathbb{Z}/5)^4$, $(F_4, \frac{2}{3}C)$ giving $H_2(L, \mathbb{Z}) = (\mathbb{Z}/3)^6$, and $(F_5, \frac{2}{3}C)$ giving $H_2(L, \mathbb{Z}) = (\mathbb{Z}/3)^8$. $(F_3, \frac{3}{4}C)$ is not orbifold simply connected, so for $H_2(L, \mathbb{Z}) = (\mathbb{Z}/4)^4$ we need another example. It is given by $(Q, \frac{3}{4}C)$ where $Q \subset \mathbb{P}^3$ is the quadric cone and $C \subset Q$ is a smooth degree 5 curve (thus it has to pass through the vertex). The corresponding Seifert bundle can also be realized as the link of the singularity $x^5 + y^5 + xz^2 + u^4 = 0$.

For examples with $H_2(L, \mathbb{Z}) = (\mathbb{Z}/m)^2$ it is probably easiest to use a general degree 1 smooth Del Pezzo surface S and a smooth elliptic member of $|-K_S|$. The conditions of (74) are very easy to check for any $m \geq 2$.

Let \mathbb{F}_n be the minimal ruled surface with a negative section $E \subset \mathbb{F}_n$ with $E^2 = -n$ and fiber F . Take a smooth curve $C \subset |2E + (2n+3)F|$ which is transversal to E . Then $g(C) = n+2$ and $(\mathbb{F}_n, (1 - \frac{1}{2})C + (1 - \frac{1}{m})E)$ is a log Del Pezzo surface

for $m > 2n$. The conditions of (74) again present no problems for $m \geq 7$. These give examples with $H_2(L, \mathbb{Z}) = (\mathbb{Z}/2)^{2n}$ for any $n \geq 2$.

Finally, an example with $H_2(L, \mathbb{Z}) = (\mathbb{Z}/3)^4$ can be obtained from $(\mathbb{F}_1, \frac{2}{3}D + \frac{1}{2}E)$ where $D \in |2E + 4F|$. \square

84 (Proof of (6)). By (78), the existence of such Seifert bundles is reduced to a question about log Del Pezzo surfaces $(S, (1 - \frac{1}{m})D)$ where $m \geq 12$ and $g(D) = 1$. These are classified in (56). We get 4 cases:

- (1) $(\mathbb{P}^2, (1 - \frac{1}{m})D)$ where D is a smooth cubic. We also need $3 \nmid m$ by (78.1.c).
- (2) $(Q, (1 - \frac{1}{m})D)$ where $Q \subset \mathbb{P}^3$ is a quadric cone and D is its intersection with a quadric. We also need $2 \nmid m$ by (78.1.c).
- (3) $(\mathbb{P}(1, 2, 3), (1 - \frac{1}{m})D)$ where $D \in |\mathcal{O}(6)|$ is smooth. $[D]$ is 6 times the generator in $\text{Weil}(S)$, thus we also need m to be relatively prime to 6 by (78.1.c).
- (4) $(S_5, (1 - \frac{1}{m})D)$ with an embedding $S_5 \subset \mathbb{P}^5$ as a degree 5 surface and D is a smooth hyperplane section. $[D]$ is 5 times the generator in $\text{Weil}(S)$, thus we also need $5 \nmid m$ by (78.1.c).

The existence of Kähler–Einstein metrics is established in (76.1–4). \square

85. Let us compute $\pi_1^{orb}(S, \Delta)$ for each of the 4 cases listed in (84).

Let $\ell \subset S$ be the line, then $D \sim n \cdot \ell$ where $n = 3, 4, 6$ or 5 . Thus $\mathcal{O}_S(n \cdot \ell) \cong \mathcal{O}_S(D)$ gives an n -sheeted ramified cyclic cover $h : T \rightarrow S$ and $K_T = h^*(K_S + (1 - \frac{1}{n})D)$. Thus we get that $-K_T$ is ample and $K_T^2 = 3, 2, 1, 1$ in these 4 cases. T is a smooth Del Pezzo surface and $C := \text{red } h^{-1}(D)$ is a smooth elliptic curve.

If $C \subset \mathbb{P}^2$ is a smooth cubic then $\pi_1(\mathbb{P}^2 \setminus C) = \mathbb{Z}/3$ but as soon as we blow up at least one point on C , the surface will contain a line intersecting C in one point only. Thus $\pi_1(T \setminus C) = 1$ in all 4 cases and we conclude that $\pi_1^{orb}(\mathbb{P}^2 \setminus D) = \mathbb{Z}/3$, $\pi_1^{orb}(Q \setminus D) = \mathbb{Z}/4$, $\pi_1^{orb}(\mathbb{P}(1, 2, 3) \setminus D) = \mathbb{Z}/6$ and $\pi_1^{orb}(S_5 \setminus D) = \mathbb{Z}/5$.

Our restrictions on m specify that it be relatively prime to the order of the corresponding π_1^{orb} , thus $\pi_1^{orb}(S, \Delta) = 1$ in all 4 cases.

86 (Proof of (8)). As before, (55) reduces the first part to the enumeration of all Del Pezzo surfaces S with cyclic quotient Du Val singularities such that $H_1^{orb}(S) = 0$. This is accomplished in (70). Kähler–Einstein metrics are constructed in (77).

The second part of (8) follows from (62) and (76.5).

This completes the proof of (8). \square

9. LINKS OF LOG TERMINAL SINGULARITIES

Let $f : Y \rightarrow (X, \Delta)$ be a Seifert \mathbb{C}^* -bundle. One can naturally compactify it by adding a zero and infinity section, see [Kol04b, 14]. The infinity section is contractible to a singular point iff $c_1(Y/X)$ is ample. As noted by [Pin77, Dem88], this establishes an equivalence between Seifert \mathbb{C}^* -bundles with $c_1(Y/X)$ ample and singularities with a good \mathbb{C}^* -action.

The canonical class of this singularity is \mathbb{Q} -Cartier iff the orbifold canonical class is a rational multiple of $c_1(Y/X)$. Furthermore, the generalized adjunction formula (cf. [Kol92, Sec.16]) implies that the singularity is log terminal iff (X, Δ) is log terminal and $-(K_X + \Delta)$ is ample.

Thus we have established an equivalence between:

- (1) Smooth Seifert \mathbb{C}^* -bundles $f : Y \rightarrow (X, \Delta)$ such that $-(K_X + \Delta)$ is ample and $c_1(Y/X)$ is a positive multiple of $-(K_X + \Delta)$.
- (2) Isolated log terminal singularities with a good \mathbb{C}^* -action.

It is natural to ask, to what extent the results of this note generalize to arbitrary log terminal singularities.

Problem 87. Let $0 \in X$ be an isolated log terminal singularity with link M .

- (1) Is $\pi_1(M)$ finite?
- (2) In dimension 3, do the restrictions of (4) also apply to M ?
- (3) Is there any connection between the log terminality of X and the existence of positive Ricci curvature metrics on M ?
- (4) Can one obtain Einstein metrics on links without \mathbb{C}^* -action?

Log canonical singularities with \mathbb{C}^* -action lead to Seifert \mathbb{C}^* -bundles whose base is a Calabi–Yau orbifold:

Proposition 88. *Let $f : L \rightarrow (X, \sum(1 - \frac{1}{m_i})D_i)$ be a Seifert bundle, L smooth. Assume that $H_1(L, \mathbb{Z}) = 0$ and $(X, \sum(1 - \frac{1}{m_i})D_i)$ is a Calabi–Yau orbifold, that is, $K_X + \sum(1 - \frac{1}{m_i})D_i$ is numerically trivial.*

Then $\sum(1 - \frac{1}{m_i})D_i = 0$ and X is a Calabi–Yau orbifold with trivial canonical class.

Proof. If $\sum(1 - \frac{1}{m_i})D_i \neq 0$, pick a prime p dividing at least one of the m_i . Let us clear denominators in $\sum(1 - \frac{1}{m_i})D_i \equiv -K_X$ and rewrite this as

$$\sum_{i:p|m_i} a_i D_i \sim p \cdot (\text{integral Weil divisor}),$$

where not all the a_i are divisible by p . Thus by (37) we get that $H_1^{orb}(X, \sum(1 - \frac{1}{m_i})D_i) \neq 0$, a contradiction to $H_1(L, \mathbb{Z}) = 0$.

Thus the canonical class is torsion. Thus it gives a torsion element of $H^2(X \setminus \text{Sing } X, \mathbb{Z})$ which is, however, torsion free since $H_1(X \setminus \text{Sing } X, \mathbb{Z}) = 0$. Thus the canonical class is trivial. \square

In dimension 5, one can be even more precise, as conjectured by Galicki:

Corollary 89. *Let $f : L \rightarrow (S, \sum(1 - \frac{1}{m_i})D_i)$ be a 5-dimensional Seifert bundle, L smooth. Assume that $H_1(L, \mathbb{Z}) = 0$ and $(S, \sum(1 - \frac{1}{m_i})D_i)$ is a Calabi–Yau orbifold. Then*

- (1) *the minimal resolution of S is a K3-surface, and*
- (2) *L is homeomorphic to the connected sum of at most 21 copies of $S^2 \times S^3$.*

Proof. By (88), S is a simply connected orbifold with trivial canonical class. In dimension 2 cyclic quotient singularities with trivial canonical class are Du Val. Let $h : S' \rightarrow S$ be the minimal resolution. Then $K_{S'} = h^*K_S$ and $H_1(S', \mathbb{Z}) = 0$. Therefore S' is a K3 surface and $\text{rank } H^2(S, \mathbb{Q}) \leq \text{rank } H^2(S', \mathbb{Q}) = 22$, the maximum achieved in the smooth case only.

(It may happen that S' is a K3 surface but $\pi_1^{orb}(S) \neq 1$. A famous example is the Kummer surface, a quotient of an Abelian surface by $p \mapsto -p$. In this case $\pi_1^{orb}(S)$ sits in an exact sequence

$$0 \rightarrow \mathbb{Z}^4 \rightarrow \pi_1^{orb}(S) \rightarrow \mathbb{Z}/2 \rightarrow 0.$$

There are numerous similar cases, classified by [Muk88].)

Since there is no branch divisor, (50) shows that $H_2(L, \mathbb{Z})$ is torsion free. The second Stiefel–Whitney class is zero by (90). Therefore L is homeomorphic to the connected sum of at most 21 copies of $S^2 \times S^3$ by (3). \square

The following lemma is essentially in [Mor97, BGN03a].

Lemma 90. *Let $f : L \rightarrow (X, \Delta)$ be a Seifert bundle. Assume that $H_1(L, \mathbb{Z}) = 0$ and that $K_X + \Delta$ is a rational multiple of $c_1(L/X)$. Then $w_2(L) = 0$.*

Proof. L is an orientable hypersurface in the corresponding Seifert \mathbb{C}^* -bundle $Y \rightarrow S$ and $w_2(L) = c_1(Y)|_L \pmod{2}$. By [Kol04b, 41 and 16],

$$c_1(Y) = -K_Y = f^*(K_X + \Delta) = c \cdot f^*c_1(Y/X) = 0$$

in rational cohomology. Since $H_1(L, \mathbb{Z}) = 0$, the second cohomology has no torsion thus $c_1(Y) = 0$ and also $w_2(L) = 0$. \square

REFERENCES

- [BG00] Charles P. Boyer and Krzysztof Galicki, *On Sasakian-Einstein geometry*, Internat. J. Math. **11** (2000), no. 7, 873–909. MR **2001k**:53081
- [BG01] ———, *New Einstein metrics in dimension five*, J. Differential Geom. **57** (2001), no. 3, 443–463. MR **2003b**:53047
- [BG03] ———, *New Einstein metrics on $8\#(S^2 \times S^3)$* , Differential Geom. Appl. **19** (2003), no. 2, 245–251. MR **2002**:662
- [BG04a] ———, *Einstein metrics on rational homology spheres*.
- [BG04b] ———, *Sasakian geometry, hypersurface singularities and Einstein metrics*.
- [BGK04] Charles P. Boyer, Krzysztof Galicki, and János Kollár, *Einstein metrics on spheres*, Ann. of Math. (2004).
- [BGN02] Charles P. Boyer, Krzysztof Galicki, and Michael Nakamaye, *Sasakian-Einstein structures on $9\#(S^2 \times S^3)$* , Trans. Amer. Math. Soc. **354** (2002), no. 8, 2983–2996 (electronic). MR **2003g**:53061
- [BGN03a] ———, *On positive Sasakian geometry*, Geom. Dedicata **101** (2003), no. 1, 93–102.
- [BGN03b] ———, *On the geometry of Sasakian-Einstein 5-manifolds*, Math. Ann. **325** (2003), no. 3, 485–524. MR **2004b**:53061
- [BHPvdV04] Wolf P. Barth, Klaus Hulek, Chris A. M. Peters, and Antonius Van de Ven, *Compact complex surfaces*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 4, Springer-Verlag, Berlin, 2004. MR **2004**:225
- [Bri66] Egbert Brieskorn, *Beispiele zur Differentialtopologie von Singularitäten*, Invent. Math. **2** (1966), 1–14. MR **34**:6788
- [Cam91] F. Campana, *Une version géométrique généralisée du théorème du produit de Nadel*, Bull. Soc. Math. France **119** (1991), no. 4, 479–493. MR **93h**:14029
- [Dem88] Michel Demazure, *Anneaux gradués normaux*, Introduction à la théorie des singularités, II, Travaux en Cours, vol. 37, Hermann, Paris, 1988, pp. 35–68. MR **91k**:14004
- [DK01] Jean-Pierre Demailly and János Kollár, *Semi-continuity of complex singularity exponents and Kähler-Einstein metrics on Fano orbifolds*, Ann. Sci. École Norm. Sup. (4) **34** (2001), no. 4, 525–556. MR **2002e**:32032
- [Dol75] I. V. Dolgachev, *Automorphic forms, and quasihomogeneous singularities*, Funkcional. Anal. i Priložen. **9** (1975), no. 2, 67–68. MR **58**:27958
- [Fur86] Mikio Furushima, *Singular del Pezzo surfaces and analytic compactifications of 3-dimensional complex affine space \mathbb{C}^3* , Nagoya Math. J. **104** (1986), 1–28. MR **88m**:32051
- [FZ03] Hubert Flenner and Mikhail Zaidenberg, *Normal affine surfaces with \mathbb{C}^* -actions*, Osaka J. Math. **40** (2003), no. 4, 981–1009. MR **2002**:670
- [GH78] Phillip Griffiths and Joseph Harris, *Principles of algebraic geometry*, Wiley-Interscience [John Wiley & Sons], New York, 1978. MR **80b**:14001

- [Hen11] Archibald Henderson, *The twenty-seven lines upon the cubic surface*, Cambridge Tracts in Mathematics and Mathematical Physics, Cambridge Univ. Press, 1911. MR 22 #9905
- [HS91] A. Haefliger and É. Salem, *Actions of tori on orbifolds*, Ann. Global Anal. Geom. **9** (1991), no. 1, 37–59. MR 92f:57047
- [JK01] J. M. Johnson and J. Kollár, *Kähler-Einstein metrics on log del Pezzo surfaces in weighted projective 3-spaces*, Ann. Inst. Fourier (Grenoble) **51** (2001), no. 1, 69–79. MR 2002b:32041
- [KM98] János Kollár and Shigefumi Mori, *Birational geometry of algebraic varieties*, Cambridge Tracts in Mathematics, vol. 134, Cambridge University Press, Cambridge, 1998, With the collaboration of C. H. Clemens and A. Corti, Translated from the 1998 Japanese original. MR 2000b:14018
- [KM99] Seán Keel and James McKernan, *Rational curves on quasi-projective surfaces*, Mem. Amer. Math. Soc. **140** (1999), no. 669, viii+153. MR 99m:14068
- [KMM92] János Kollár, Yoichi Miyaoka, and Shigefumi Mori, *Rational connectedness and boundedness of Fano manifolds*, J. Differential Geom. **36** (1992), no. 3, 765–779. MR 94g:14021
- [Kob63] Shoshichi Kobayashi, *Topology of positively pinched Kaehler manifolds*, Tôhoku Math. J. (2) **15** (1963), 121–139. MR 27 #4185
- [Kol92] János Kollár (ed.), *Flips and abundance for algebraic threefolds*, Paris, Société Mathématique de France, 1992, Papers from the Second Summer Seminar on Algebraic Geometry held at the University of Utah, Salt Lake City, Utah, August 1991, Astérisque No. 211 (1992). MR 94f:14013
- [Kol94] ———, *Log surfaces of general type; some conjectures*, Classification of algebraic varieties (L'Aquila, 1992), Contemp. Math., vol. 162, Amer. Math. Soc., Providence, RI, 1994, pp. 261–275. MR 95c:14042
- [Kol04a] ———, *Einstein metrics on connected sums of $S^2 \times S^3$* , to appear (2004).
- [Kol04b] ———, *Seifert G_m -bundles*, to appear (2004).
- [Miy01] Masayoshi Miyanishi, *Open algebraic surfaces*, CRM Monograph Series, vol. 12, American Mathematical Society, Providence, RI, 2001. MR 2002e:14101
- [MM86] Yoichi Miyaoka and Shigefumi Mori, *A numerical criterion for uniruledness*, Ann. of Math. (2) **124** (1986), no. 1, 65–69. MR 87k:14046
- [Mor97] Andrei Moroianu, *Parallel and Killing spinors on Spin^c manifolds*, Comm. Math. Phys. **187** (1997), no. 2, 417–427. MR 98i:58245
- [Muk88] Shigeru Mukai, *Finite groups of automorphisms of K3 surfaces and the Mathieu group*, Invent. Math. **94** (1988), no. 1, 183–221. MR 90b:32053
- [Mum61] David Mumford, *The topology of normal singularities of an algebraic surface and a criterion for simplicity*, Inst. Hautes Études Sci. Publ. Math. (1961), no. 9, 5–22. MR 27 #3643
- [MZ88] M. Miyanishi and D.-Q. Zhang, *Gorenstein log del Pezzo surfaces of rank one*, J. Algebra **118** (1988), no. 1, 63–84. MR 89i:14033
- [MZ93] ———, *Gorenstein log del Pezzo surfaces. II*, J. Algebra **156** (1993), no. 1, 183–193. MR 94m:14045
- [Nad90] Alan Michael Nadel, *Multiplier ideal sheaves and Kähler-Einstein metrics of positive scalar curvature*, Ann. of Math. (2) **132** (1990), no. 3, 549–596. MR 92d:32038
- [Nad91] ———, *The boundedness of degree of Fano varieties with Picard number one*, J. Amer. Math. Soc. **4** (1991), no. 4, 681–692. MR 93g:14048
- [OW75] Peter Orlik and Philip Wagreich, *Seifert n -manifolds*, Invent. Math. **28** (1975), 137–159. MR 50 #13596
- [Pin77] H. Pinkham, *Normal surface singularities with C^* action*, Math. Ann. **227** (1977), no. 2, 183–193. MR 55 #5623
- [Sak84] Fumio Sakai, *Anticanonical models of rational surfaces*, Math. Ann. **269** (1984), no. 3, 389–410. MR 85m:14058
- [Sco83] Peter Scott, *The geometries of 3-manifolds*, Bull. London Math. Soc. **15** (1983), no. 5, 401–487. MR 84m:57009
- [Sei32] Herbert Seifert, *Topologie dreidimensionaler gefaserte Räume*, Acta Math. **60** (1932), 148–238.

- [Sho00] V. V. Shokurov, *Complements on surfaces*, J. Math. Sci. (New York) **102** (2000), no. 2, 3876–3932. MR **2002c**:14030
- [Sma62] Stephen Smale, *On the structure of 5-manifolds*, Ann. of Math. (2) **75** (1962), 38–46. MR 25 #4544
- [Thu78] W. Thurston, *The geometry and topology of 3-manifolds*, Princeton University, Mimeographed Notes, 1978.
- [Yau77] Shing Tung Yau, *Calabi's conjecture and some new results in algebraic geometry*, Proc. Nat. Acad. Sci. U.S.A. **74** (1977), no. 5, 1798–1799. MR 56 #9467

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